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Efficiency in Partnerships*

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We provide the necessary and sufficient condition for a partnership to be able to sustain efficiency when the output is stochastic. When limited liability is imposed, we show that only the level of the average liability of the partnership is important; the individual levels of liability are irrelevant. However, the allocation of the total liability among the partners is crucial when additional conditions such as neutrality or individual rationality are imposed. *Journal of Economic Literature Classification Numbers*: 022, 026, 610.

1. INTRODUCTION

A partnership is characterized by joint ownership and by sharing of the final output among the partners. The early literature on partnerships (Alchian and Demsetz [1], Holmström [9], Radner *et al.* [12]) has argued that partnerships are inefficient forms of organization because the partners cannot solve their moral hazard problem. In contrast, recent results (Williams and Radner [14], Legros [10], Matsushima [11]) challenge this conclusion and prove that there exist nontrivial environments in which partnerships are efficient: moral hazard does not always conflict with efficiency when there is joint production. This paper is in the latter tradition. We analyze the efficiency properties of a partnership when

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the output is stochastic and when side-payments are allowed and we are able to give the necessary and sufficient condition for sustaining efficiency by balanced transfer rules.

The basic model is the following. Utilities are assumed to be fully transferable. The players simultaneously choose actions that are not verifiable. These actions generate an output, possibly a stochastic variable, whose realization becomes public information. After the realization of the output, each partner receives a monetary transfer following a previously agreed transfer rule. Williams and Radner [14] is the first paper to argue that when the output is stochastic it might be possible to attain the first best. They give sufficient conditions for this to happen. However, their sufficient conditions seem to be quite strong and it is somehow difficult to interpret them. When the output is nonstochastic, Legros [10] presents a necessary and sufficient condition for attaining efficiency. More importantly, a type of "static folk theorem" is proved in that work: for any positive ε, ε-efficiency can be sustained in mixed strategies. In a stochastic environment, Matsushima [11] gives a necessary and sufficient condition for the existence of a special class of transfer rules, called "penalty rules," that sustain efficiency. We provide in Section 4 of this paper a necessary and sufficient condition for attaining efficiency when the output is stochastic under the assumption that the action spaces are finite. The interpretation of this condition is intuitive and is similar in spirit to Legros [10] and Matsushima [11].

We do not restrict the set of transfer rules that the partners can choose from. It follows that a transfer rule that sustains efficiency might require a player to pay a large amount to the other players for some realizations of the output. In Section 5, we introduce bankruptcy constraints and we impose that the share of each partner is bounded below. The liability of a partner is the maximum sum that she can be asked to pay. The average liability in the partnership is equal to the ratio of the total liability (the sum of the individual liabilities) to the number of partners. We show that only the average liability matters for the existence of a sharing rule that implements the first best subject to limited liability constraints. It is necessary and sufficient that this average liability be greater than a certain index. This index expresses the likelihood of deviations in the partnership and is equal to the ratio of the average variation in expected utility levels of the partners to a measure of the closeness of deviations. We formalize in Section 3 this measure of closeness of strategies.

In Section 6, we define a transfer rule to be neutral if the expected (with respect to the efficient probability distribution) transfer to each partner is zero (this corresponds to the expected budget balanced condition of d'Aspremont and Gérard-Varet [3]). It is natural to require neutrality since with neutral transfer rules, we have implementation of the vector of efficient actions and of the vector of expected utilities at the efficient actions. The implementation of the vector of utility levels is relevant when the utility levels are the outcome of some ex-ante bargaining or correspond to some concept of fairness. If balanced transfer rules that sustain efficiency exist, then there always exist neutral transfer rules that sustain efficiency. However, with limited liability, it can be the case that no neutral transfer rule exists, even if there exist balanced transfer rules that sustain efficiency and that are consistent with limited liability. We give a necessary and a sufficient condition for the existence of a neutral transfer rule with limited liability. When neutrality and limited liability are jointly imposed, the partition of the total liability among the partners is crucial for the existence of balanced transfer rules.

When individual rationality—a participation constraint—is imposed in conjunction with limited liability, two facts matter for the existence of a solution to the partnership problem: the relative liability levels and the first best opportunity costs (i.e., the difference between the first best and the reservation utility levels of the partners). If the first best opportunity costs are positive for all the partners, then neutrality and limited liability imply individual rationality and limited liability; otherwise, neutral transfer rules can fail to be individually rational.

The rest of the paper is organized as follows. We present the model in the next section. In Section 3, we introduce a measure of closeness of strategies that is related to an "extended metric" on the space of probability measures over the set of output levels. In Section 4, we characterize the environments for which there exist balanced transfer rules that sustain efficiency. We study in Section 5 the question of limited liability. We restrict in Sections 6 and 7 the transfer rules to be neutral and individually rational respectively and we derive the corresponding conditions for the sustainability of efficiency under these additional restrictions. We present an example in Section 8 and we conclude in Section 9.

2. The Model

The model that we consider in this paper is the following. There is a finite set $N = \{1, ..., n\}$ of players (or partners) and each partner i can take an action in the finite set $A_i = \{a_i(1), ..., a_i(T_i)\}$. The joint actions of the partners induce a stochastic output whose realizations lie in the finite set $\Omega = \{w_1, ..., w_l\}$. Actions and outputs can be multidimensional. (For instance, an output w can be defined in terms of quantity and quality.) For each vector of joint actions $a \in A = \mathsf{X}_{i \in N} A_i$, there exists a probability distribution over the set Ω of possible outputs. $\pi(w; a)$ is the probability that w is realized when the vector of joint actions is a. $\pi[a]$ denotes the vector $(\pi(w_1; a), \ldots, \pi(w_l; a))$. $\pi[a]$ satisfies

$$\forall w \in \Omega, \forall a \in A, \qquad \pi(w; a) \geqslant 0,$$

$$\sum_{w \in \Omega} \pi(w; a) = 1.$$

We suppose that the partners have quasi-linear utility functions, i.e., side payments are allowed. If partner i receives an amount of money of m while output w is realized and while the vector of joint actions is a, her utility will be $m + u_i(w, a)$.[1] Let $U_i(a) \equiv \sum_{w \in \Omega} \pi(w; a) \cdot u_i(w, a)$ be the (expected) direct utility of player i when the vector of joint actions is a. Quasi-linear utility functions are used extensively in the public economics and in the incentive literatures and have the nice property of freeing the analysis of income effects (e.g., Groves [7], Green and Laffont [8]).

A *pure strategy* for player i is a choice of actions a_i in A_i. A *mixed strategy* for player i is a probability distribution α_i over the set of actions. Let M_i be the set of possible mixed strategies for player i and let M be the Cartesian product $\mathsf{X}_i M_i$. Then $\forall \alpha_i \in M_i$, $\forall a_i \in A_i$, $\alpha_i(a_i) \geqslant 0$ and $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$. We will use the following notation:

$$A_{-i} \equiv \mathop{\mathsf{X}}_{j \neq i} A_j.$$

$$\forall a \in A, \forall \hat{a}_i \in A_i, \qquad a \backslash \hat{a}_i \equiv (a_{-i}, \hat{a}_i).$$

$$\forall a \in A, \forall \alpha_i \in M_i, \qquad \pi(w; a \backslash \alpha_i) \equiv \sum_{\hat{a}_i \in A_i} \alpha_i(\hat{a}_i) \cdot \pi(w; a \backslash \hat{a}_i).$$

A *transfer rule* is a function $t: \Omega \to \mathbb{R}^n$, i.e., $t(w)$ is an n-vector $(t_1(w), \ldots, t_n(w))$, where $t_i(w)$ is the transfer (possibly negative) to partner i if the output w is realized. A transfer rule is said to be *balanced* if

$$\forall w \in \Omega, \qquad \sum_{i \in N} t_i(w) = 0.$$

It is possible to identify the set of budget balanced transfer rules with the set of maps from Ω to $\mathbb{R}^{n-1}$. In this case, the transfer to partner n if output w is observed is $-\sum_{i \neq n} t_i(w)$.

[1] A more general form of quasi-linear utility function for partner i is $c_i \cdot m + u_i(w, a)$, where c_i is a positive constant. We suppose here that $c_i = 1$ for each partner i. It is clear that the efficient vector of actions depends on the c_i's. Nevertheless, incentive compatibility is preserved when partner i's utility function is divided by c_i. Consequently, the results of this paper can be generalized to the more general utility functions.

The expected utility of player i when the vector of joint actions is a and the transfer rule is t is

$$EU_i(a, t) = \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a) + U_i(a).$$

A vector of joint (pure) strategies a is a *Nash equilibrium of the game induced by the transfer rule t* if for every $i \in N$ and every $\alpha_i \in M_i$,

$$EU_i(a, t) \geqslant EU_i(a \backslash \alpha_i, t).$$

A transfer rule t is said to *sustain* a vector of joint strategies a if a is a Nash equilibrium in the game induced by t.

By finiteness of the sets of actions and of output levels, there exists a vector of joint actions a^* such that

$$a^* \in \operatorname*{Argmax}_{a \in A} \sum_{i \in N} U_i(a).$$

Such a vector of joint actions is said to be *efficient.* The reader will note that the sum of utilities is the only efficiency criterion when the utility is transferable.

The *partnership's problem* is to find a balanced transfer rule t such that a^* is a pure strategy Nash equilibrium of the game with strategy spaces $\{A_i\}$ and with payoff functions $\{EU_i(a, t)\}$. The partnership problem has a *solution* t if the following incentive compatibility conditions are satisfied:

$$\forall i \in N, \forall a_i \in A_i, \qquad EU_i(a^* \backslash a_i, t) \leqslant EU_i(a^*, t).$$

3. A Measure of Closeness of Strategies

For the purpose of sustaining a^*, the physical proximity of the strategies is irrelevant; what matters is the proximity of the probability measures on Ω induced by the strategies.[2] For instance, if for all a, $\pi[a] = \pi$, then for any two profiles of strategies α and $\hat{\alpha}$, $\pi[\alpha] = \pi[\hat{\alpha}]$, i.e., while the two strategy profiles can be far apart (e.g., with respect to the Euclidian distance on the space M) they induce the same probability measure on Ω. In this respect, they have the same influence on the expected transfers of the players.

[2] This idea is not new. For instance, Fudenberg and Levine [5] propose a utility based metric that generalizes Wald's [13] concept of "intrinsic distance." Our measure of distance between strategies is based on the distributions over outputs that are generated by the strategies; it cannot be based on the utility functions because the transfer rule (that determines the utility functions) is a variable of the model.

This section develops a measure of closeness of strategies that is based on this observation. This section is technical and can be ignored in a first reading. Let $\Delta(\Omega)$ be the set of probability measures on Ω. I.e., $\Delta(\Omega) \equiv \{q : \Omega \to [0, 1] \mid \sum_{w \in \Omega} q(w) = 1\}$. We define an operator on $\Delta(\Omega)$ as follows. For any subset Q of $\Delta(\Omega)$, let

$$d(Q) \equiv 1 - \sum_{w \in \Omega} \inf_{q \in Q} q(w).$$

$d(Q)$ measures the distance between the elements of Q. We show in the following lemma that d is a generalization of the concept of metric and that d provides a metric on $\Delta(\Omega)$ when d is restricted to subsets of $\Delta(\Omega)$ having only two elements.

Remark. We will see in Lemma 1 that $d(Q) = 0$ if, and only if, Q is empty or is a singleton. It can be showed that $d(Q)$ is maximum, i.e., is equal to one if, and only if, the intersection of the closure of Q with each face $F_w = \{q \in \Delta(\Omega) \mid q(w) = 0\}$ of $\Delta(\Omega)$, where $w \in \Omega$, is nonempty. Indeed, in this case for each $w \in \Omega$ there exists a sequence $\{q^k\} \subset Q$ such that $\lim_{k \to \infty} q^k(w) = 0$. In particular, $d(\Delta(\Omega)) = 1$, and if $\partial\Delta(\Omega)$ is the boundary of the simplex, $d(\partial\Delta(\Omega)) = 1$. Let $\Omega = \{0, 1\}$, and let $Q = \{(1/n, 1 - 1/n) \mid n \in \mathbb{Z}_+ \setminus \{0\}\}$; then $d(Q) = 1$ since the extreme points $(0, 1)$ and $(1, 0)$ belong to the closure of Q (note that $(0, 1)$ does not belong to Q). If the closure of Q does not intersect some face F_w, then $d(Q)$ is strictly between 0 and 1. For instance, if $\Omega = \{0, 1\}$, $Q^0 = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{3}{4}, \frac{1}{4})\}$, $Q^1 = \{(\frac{1}{3}, \frac{2}{3}), (\frac{3}{4}, \frac{1}{4})\}$, then $d(Q^0) = d(Q^1) = \frac{5}{12}$.

LEMMA 1. *The operator d has the following properties.*

(i) $d(Q) \geqslant 0$ *for any subset Q of $\Delta(\Omega)$.*

(ii) $[d(Q) = 0] \Leftrightarrow [|Q| = 1]$ *for any nonempty subset Q of $\Delta(\Omega)$.*

(iii) $d(Q \cup T) + d(T \cup V) \geqslant d(Q \cup V)$ *for any three subsets Q, T and V of $\Delta(\Omega)$.*

(iv) *Let $\forall q, q' \in \Delta(\Omega)$, $\tilde{d}(q, q') \equiv d(\{q, q'\})$. Then $\tilde{d}$ is a metric for the space $\Delta(\Omega)$.*

(v) $[Q \subset T] \Rightarrow [d(Q) \leqslant d(T)]$ *for any subsets Q and T of $\Delta(\Omega)$.*

Proof. See the Appendix. ▌

It is now possible to propose a measure ρ of closeness of strategies. Recall that M_i is the set of mixed strategies for player i and that $M = \times_{i \in N} M_i$. Let $\alpha = (\alpha_1, ..., \alpha_n) \in M$ be a vector of strategies. We consider the probability measure π and the vector of actions a^*. Each strategy α_i generates a probability distribution $\pi[a^* \backslash \alpha_i]$. For any $\eta \in [0, 1]$, we define the strategies α_i to be η-close with respect to (π, a^*) if the d-distance between the elements of the set $\{\pi[a^*\backslash\alpha_i] \mid i \in N\}$ is equal to η. Hence, our measure of closeness of strategies is given by the operator ρ;

$$\forall \alpha \in M, \qquad \rho(\alpha) \equiv d(\{\pi[\omega; a^*\backslash\alpha_i] \mid i \in N\}). \tag{1}$$

It is possible to interpret $\rho(\alpha)$ as a measure of closeness of strategies because the underlying operator d that defines ρ has properties that generalize those of a metric. By definition of d, it is straightforward that (1) is equivalent to

$$\forall \alpha \in M, \qquad \rho(\alpha) = 1 - \sum_{w \in \Omega} \min_{i \in N} \pi(w; a^*\backslash\alpha_i).$$

4. Characterization

For each vector of mixed strategies α, we can define an index $\beta(\alpha)$ by

$$\beta(\alpha) = \begin{cases} \dfrac{\sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*))}{n \cdot \rho(\alpha)} & \text{if } \rho(\alpha) \neq 0 \\ & \text{or } \sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*)) \neq 0. \\ = 0 & \text{otherwise.}^3 \end{cases}$$

This index is the average gain from deviations $\sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*))/n$ normalized by the measure of closeness of the deviations. We want to interpret β as a measure of the likelihood of a deviation. In short, the higher is β, the higher are the incentives for deviations. The first result of this paper states that there exists a solution to the partnership problem if, and only if, $\beta(\alpha)$ is bounded above by a finite nonnegative number. Below, β^* denotes the supremum of the set $\{\beta(\alpha) \mid \alpha \in M\}$.

THEOREM 2. *There exists a bounded balanced transfer rule t that sustains a^* if, and only if, $\beta^* < +\infty$.*

Proof. See the Appendix. ▮

This condition is a generalization of the characterization by Legros [10] in the deterministic case. We would like to suggest an intuition for the necessity of this result. Two cases are possible for a given deviation. Suppose first that the average gain from a deviation α is nonpositive. In this case, there exists a vector s such that $\sum_{i \in N} s_i = 0$ and such that for

[3] This is in order for β to be a well defined operator. The choice of zero is convenient but otherwise arbitary.

all i, $U_i(a^*\backslash\alpha_i) \leqslant U_i(a^*) + s_i$. Consequently, it is possible to find a transfer rule that will prevent each player i from deviating from a_i^* to α_i. (Choose t such that $\forall i \in N$, $\sum_{w \in \Omega} t_i(w) \cdot (\pi(w; a^*) - \pi(w; a^*\backslash\alpha_i)) = s_i$. It is easy to see that such a t exists.) Hence, even if $\rho(\alpha) = 0$, i.e., if all the players can induce the same distribution on Ω by deviating, local punishments are possible. Suppose now that the average gain from a deviation α is positive. In this case, it is necessary that $\rho(\alpha)$ be positive, i.e., that at most $n-1$ players can induce the same distribution on Ω by deviating from a^*, for otherwise "local punishments" are not possible. Indeed, if $\rho(\alpha) = 0$, then, by balancedness of t,

$$\begin{aligned}
&\sum_{w \in \Omega} t_n(w) \cdot (\pi(w; a^*) - \pi(w; a^*\backslash\alpha_n)) \\
&\qquad = -\sum_{i \neq n} \sum_{w \in \Omega} t_i(w) \cdot (\pi(w; a^*) - \pi(w; a^*\backslash\alpha_n)) \\
&\qquad = -\sum_{i \neq n} \sum_{w \in \Omega} t_i(w) \cdot (\pi(w; a^*) - \pi(w; a^*\backslash\alpha_i)).
\end{aligned}$$

Let $s_i = \sum_{w \in \Omega} t_i(w) \cdot (\pi(w; a^*) - \pi(w; a^*\backslash\alpha_i))$. Then the local incentive compatibility conditions imply that $\forall i \in N$, $U_i(a^*\backslash\alpha_i) \leqslant U_i(a^*) + s_i$. But since $\rho(\alpha) = 0$, $\sum_{i \in N} s_i = 0$, and this contradicts the assumption that the average gain from the deviation α is positive. Consequently, $\beta^* < +\infty$ is a necessary condition for local, hence global, incentive compatibility. Theorem 2 shows that this condition is also sufficient.

Remark. It is possible to show that a^* is a strict Nash equilibrium, i.e., that for any i, a_i^* is the unique best response to a^*_{-i}, if, and only if, $\beta^* < +\infty$, and $[\alpha \neq a^*$ and $\rho(\alpha) = 0] \Rightarrow [\beta(\alpha) = -\infty]$.

There is a corollary to this theorem which is also a consequence of a result by Carver [2]. The condition identified above is sensitive to a change in the direct utility functions U_i. It is of interest to know when a partnership can sustain a given vector of actions a^* for any choice of the utility functions. (Observe that in this case a^* is not obligatorily efficient for all specifications of the utility functions.) It should be clear to the reader that, in order to attain such a strong result, the condition on the environment is a pure informational requirement: it must be true that for any deviation α, $\rho(\alpha) > 0$, i.e., that irrespective of α, all the players cannot induce the same probability distribution on Ω by deviating from a^*. This result appears in Matsushima [11] and is similar in spirit to the result obtained, in a different setting, by d'Aspremont and Gerard-Varet [3].

PROPOSITION 3. *There exists a balanced transfer rule t that sustains a^* for every $\{U_i, i \in N\}$ if, and only if, for every $\alpha \neq a^*$, $\rho(\alpha) \neq 0$.*

Proof. Follows Theorem 2. ∎

This condition is also necessary and sufficient for the existence of transfer rules having the property that the expected share of each partner is maximum when she is eficient, i.e.,

$$\forall i \in N, \forall \alpha_i \in M_i, \qquad \sum_{w \in \Omega} t_i(w) \cdot [\pi(w; a^*) - \pi(w; a^* \backslash \alpha_i)] > 0.$$

Matsushima [11] calls such transfer rules *penalty rules.*

In the deterministic case, it is assumed that for every a, there exists $w(a) \in \Omega$ such that $\pi(w(a); a) = 1$. In this case, Legros [10] presents an interpretation similar to those of Theorem 2 and Proposition 3. The condition in Proposition 3 implies, in the deterministic case, that for every $a \in A$, there exist $i \in N$ and $j \in N$ such that $w(a^* \backslash a_i) \neq w(a^* \backslash a_j)$. That is, by observing the output, it is possible to identify who did not deviate. However, the fact that it is possible to verify who did not deviate is a property that we get for free in the deterministic case because only one output level can be atteined by a given vector of actions. We show in Theorem 2 and Proposition 3 that the possibility to verify who did not deviate by observing the output is not essential in the stochastic case.

If there exists a transfer rule that sustains a^*, then there exist an infinity of such solutions. Indeed, for any t that solves the partnership problem, another transfer rule $\tilde{t}$ where $\tilde{t}_i(w) = t_i(w) + z_i$, where z is such that $\sum_{i \in N} z_i = 0$, also solves the partnership problem. However, there exists a unique solution of minimal norm, when the Euclidian norm is defined on the space $(\mathbb{R}^{|\Omega|})^n$.

PROPOSITION 4. *Suppose that there exists a solution to the partnership problem. Then there is a unique balanced transfer rule that sustains a^* whose Euclidian norm is minimum.*

Proof. Let $K \equiv \{t \mid t \text{ is balanced and sustains } a^*\}$. If there is a solution to the partnership problem, $K \neq \varnothing$. K is clearly compact and convex. Because the Euclidian norm is convex,[4] there is a unique $t \in K$ such that $\| t \| = \operatorname{argmin} \{ \| \tilde{t} \| \mid \tilde{t} \in K \}$. ∎

Theorem 2 and Proposition 3 suggest a stylized fact about partnerships: symmetry is in general bad news. Indeed, in a symmetric partnership, each partner can mimic perfectly the deviation of another partner. Hence, if there exists an action a_i for which $U_i(a^* \backslash a_i) - U_i(a^*) > 0$, there will exist a vector a such that for every $j \neq i$, $U_j(a^* \backslash a_j) - U_j(a^*) > 0$, and such that $\rho(a) = 0$. Consequently, $\beta(a) = +\infty$ and there exists no solution to the partnership problem. In this case, for any transfer rule, there will be an

[4] The norm of a Hilbert space is convex if whenever $\|x\| = 1$ and $\|y\| = 1$, then $\|x + y\| = 2$ only if $x = y$.

efficiency loss. When the output is deterministic, Legros [10] shows that the efficiency loss can be made as small as we want when the partners use mixed strategies. It is very reasonable to conjecture that the same type of approximation result will hold when the output is stochastic.

5. Limited Liability

Until now, we have been concerned with the existence of a transfer rule that will sustain a particular vector of actions. For this reason, the only restriction that we imposed on the transfer rule was a balance condition. However, in many situations, transfers are also constrained by the degree of liability of each partner. This section and the next study the consequences of a bankruptcy constraint for the existence of a solution to the partnership problem.

Let $B=(B_i)\in\mathbb{R}^n_+$, where $B_i\geqslant 0$ is the maximum liability that player i can suffer by participating in the partnership, i.e., is the maximum amount of money that she can be asked to pay. A transfer rule t is said to be *bounded with respect to* B if for every $i\in N$ and every $w\in\Omega$, $t_i(w)\geqslant -B_i$. In this section, we fix B arbitrarily. Let $d_i\colon\Omega\to\mathbb{R}_+$ and $d=(d_1, ..., d_n)$. In the same way as in the previous section, it is possible to show that there exists a balanced transfer rule that is bounded with respect to B if, and only if, for every vector of mixed strategies α, every d, and every k, either

$$\sum_{i\in N}\,[U_i(a^*\backslash\alpha_i)-U_i(a^*)]\leqslant\sum_{i\in N}B_i\cdot\sum_{w\in\Omega}d_i(w) \tag{2}$$

or

$$\exists i,\ \exists w,\qquad \pi(w;a^*\backslash\alpha_i)-d_i(w)\neq k(w). \tag{3}$$

(The proof of this claim is given in the Appendix.) From this, the following characterization result follows.

Theorem 5. *There exists a balanced transfer rule that is bounded with respect to B and that sustains a^* if, and only if,*

$$n\cdot\beta^*\leqslant\sum_{i\in N}B_i. \tag{4}$$

Proof. Fix α, d, and k arbitrarily. Suppose that neither (2) nor (3) hold, i.e.,

$$\sum_{i\in N}\,(U_i(a^*\backslash\alpha_i)-U_i(a^*))>\sum_{i\in N}B_i\cdot\sum_{w\in\Omega}d_i(w) \tag{5}$$

and

$$\forall i \in N, \forall w \in \Omega, \qquad \pi(w; a^*\backslash\alpha_i) - d_i(w) = k(w). \tag{6}$$

From (6), and using the fact that π is a probability distribution,

$$\forall i \in N, \qquad 1 - \sum_{w \in \Omega} k(w) = \sum_{w \in \Omega} d_i(w) \tag{7}$$

and

$$\forall w \in \Omega, \qquad \min_{i \in N} \pi(w; a^*\backslash\alpha_i) \geqslant k(w). \tag{8}$$

From (5), (7), and (8),

$$\begin{aligned} \sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*)) &> \sum_{i \in N} B_i \cdot \left(1 - \sum_{w \in \Omega} k(w)\right) \\ &\geqslant \sum_{i \in N} B_i \cdot \rho(\alpha) \end{aligned} \tag{9}$$

which implies that $n \cdot \beta(\alpha) > \sum_{i \in N} B_i$.

Conversely, suppose that for some α, $n \cdot \beta(\alpha) > \sum_{i \in N} B_i$. Choose

$$k(w) = \min_{i \in N} \pi(w; a^*\backslash\alpha_i)$$

and

$$d_i(w) = \pi(w; a^*\backslash\alpha_i) - \min_{j \in N} \pi(w; a^*\backslash\alpha_j).$$

Note that equalities (6) hold. By definition of d,

$$\begin{aligned} &\sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*)) - \sum_{i \in N} B_i \cdot \sum_{w \in \Omega} d_i(w) \\ &\quad = \sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*)) - \sum_{i \in N} B_i \cdot \rho(\alpha). \end{aligned}$$

Since $n \cdot \beta(\alpha) > \sum_{i \in N} B_i \geqslant 0$, obligatorily $\beta(\alpha) > 0$.

Hence, $\sum_{i \in N} (U_i(a^*\backslash\alpha_i) - U_i(a^*)) > 0$. It follows that the inequality (5) holds. This proves that there exists a balanced transfer rule that is bounded with respect to B if, and only if, for all α, $n \cdot \beta(\alpha) \leqslant \sum_{i \in N} B_i$. ∎

Theorem 5 implies that only the level of the average liability, $\sum_{i \in N} B_i/n$, matters for the existence of a balanced transfer rule that is consistent with limited liability.

6. NEUTRAL TRANSFER RULES

We can think of the transfer rule as a punishment–reward device. For this reason, it is reasonable to question the existence of transfer rules that do not punich, in an expected utility sense, the players when they are efficient. This is important if the utility functions $u_i(w, a)$ incorporate a notion of fairness or are the outcome of an ex-ante bargaining among the partners. Neutral transfer rules have the property that the ex-ante expected utility level of each partner coincides with the level that is deemed "fair." We define a transfer rule to be *neutral* if each player has a zero expected transfer at the efficient vector of actions, i.e., t is neutral when

$$\forall i \in N, \qquad \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a^*) = 0.$$

If a sharing rule t is neutral and implements a^*, then $EU_i(a^*, t) = U_i(a^*)$ for each partner $i \in N$. Consequently, neutral sharing rules implement not only the efficient vector of actions a^* but also the utility levels corresponding to the first best.

The existence of neutral and balanced transfer rules that sustain a^* is in general trivial. Indeed, consider any balanced transfer rule t that sustains a^* (hence, we suppose that the condition of Theorem 2 holds). Define a new transfer rule $\tilde{t}$ as follows,

$$\forall i \in N, \forall w \in \Omega, \qquad \tilde{t}_i(w) = t_i(w) - \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a^*).$$

Clearly, $\tilde{t}$ is neutral and is balanced. $\tilde{t}$ also sustains a^* since by adding a fixed transfer, the incentive compatibility conditions are not affected.

The reader will notice that if t is bounded with respect to B, there is no reason to expect that $\tilde{t}$ is also bounded with respect to B. For this reason, the existence of neutral and balanced transfer rules that sustain a^* and that are consistent with limited liability is not a trivial problem. Let $\bar{B}$ be the total liability available in the partnership. A *partition of this liability* among the partners is any vector $B \in \mathbb{R}^n_+$ such that $\sum_i B_i = \bar{B}$; hence a possible partition is when one partner i has full liability of $B_i = \bar{B}$ and when all the other partners have a zero liability (i.e., the other partners cannot be asked to pay a positive amount). We will show below that how the total liability is partitioned among the partners is crucial for the existence of neutral transfer rules.

The following lemma gives the necessary and sufficient condition for the existence of a neutral and balanced transfer rule that sustains a^* and that is bounded with respect to B.

LEMMA 6. *There exists a neutral and balanced transfer rule that systains a^* and that is bounded with respect to B if, and only if, there does not exist $q \in \mathbb{R}^n_+$, $a: \Omega \to \mathbb{R}$, such that*

$$\forall w, \qquad a(w) \geqslant \max_{i \in N} \left[q_i \cdot \pi(w; a^*) - (\pi(w; a^* \backslash \alpha_i) - \min_{j \in N} \pi(w; a^* \backslash \alpha_j))\right] \tag{10}$$

$$\sum_{i \in N} \left[B_i \cdot \left\{q_i - \sum_{w \in \Omega} a(w)\right\}\right] > [\bar{B} - n \cdot \beta(\alpha)] \cdot \rho(\alpha). \tag{11}$$

Proof. See the Appendix. ∎

Note that the right hand side of (11) is a measure of the "excess liability" in the partnership, weighted by the closeness of the deviations. Excess is meant with respect to the liability necessary to insure existence of a balanced transfer rule that is bounded with respect to B. This condition is not very intuitive and is not stated in terms of the primitives of the model. For this reason, we present a necessary condition and a sufficient condition that rely only on the primitives of the model.

To avoid trivialities, we suppose that the condition of Theorem 5 is satisfied, i.e., that there exists a solution to the partnership problem consistent with limited liability.

Assumption 1. The total liability $\bar{B}$ is such that there exists a balanced transfer rule that is bounded with respect to B, for any partition B of $\bar{B}$. Hence, $\forall \alpha$, $\bar{B} - n \cdot \beta(\alpha) \geqslant 0$.

We present below a necessary and a sufficient condition for the non-existence of a neutral and balanced transfer rule consistent with limited liability. These conditions rely on a comparison between the relative liability shares of the partners and the closeness of the deviations used by the players. Before stating these conditions, we need to introduce some notation. Let α be a vector of mixed strategies. We define the following sets:

$$\forall i \in N, \qquad \Omega(i; \alpha) \equiv \{w \in \Omega \mid i \in \operatorname*{argmin}_{j \in N} \pi(w; a^* \backslash \alpha_j)\}.$$

$\Omega(i; \alpha)$ is the set of outputs to which $\pi[a^* \backslash \alpha_i]$ assigns the minimal probability among all the probability measures $\pi[a^* \backslash \alpha_j]$. For any output w, and any collective deviation α, if w is an element of $\Omega(i; \alpha)$, then the difference $\pi(w, a^*) - \pi(w; a^* \backslash \alpha_i)$ is greatest for partner i. Consequently, if $t_i(w) > 0$ (resp. $t_i(w) < 0$), partner i has more to gain (resp. lose) by deviating than the other partners.

Consider the set R of all possible permutations of N, i.e.,

$$R \equiv \left\{r : N \to N \mid \forall i \in N, \forall j \in N, r(i) \neq r(j) \text{ and } \bigcup_{i \in N} \{r(i)\} = N\right\}.$$

Let i_r be the ith element of the set $r(N)$, once ordered by $\geqslant$, i.e., i_r satisfies $|\{i \in N \mid r(i) \leqslant i_r\}| = i_r$. Given a permutation r and a mixed strategy vector α, we can define the following sets:

$$F_{1_r}(r, \alpha) \equiv \Omega(r^{-1}(1_r); \alpha) \qquad (12)$$

$$F_{i_r}(r, \alpha) \equiv \Omega(r^{-1}(i_r); \alpha) \backslash F_{i_r - 1}, \qquad \text{any } i_r \geqslant 2.$$

Define $\forall i,\ p_i(r, \alpha) \equiv \sum_{w \in F_{r(i)}(r, \alpha)} \pi(w; a^*)$. If $F_{r(i)}(r, \alpha) = \varnothing$, let $p_i(r, \alpha) = 0$. Clearly, $\{F_{r(i)}(r, \alpha) \mid i \in r(N)\}$ is a partition of Ω and $\sum_{i \in N} p_i(r, \alpha) = 1$. Finally, let b_i be the *relative liability* of partner i, i.e.,

$$b_i \equiv \frac{B_i}{\bar{B}}.$$

PROPOSITION 7. *Suppose that Assumption* 1 *holds. If there does not exists a neutral and balanced transfer rule that sustains a^* and is bounded with respect to B, where B is some partition of the total liability $\bar{B}$, then there exist a vector of mixed strategies α, a permutation $r \in R$, and an index $k < n$ such that*

$$\sum_{i \leqslant k} (b_i - p_i(r, \alpha)) > 0.$$

Proof. From Lemma 6, there exist α, q, and $a(\cdot)$ such that (10) and (11) hold. Let r be a permutation of N such that $i_r \leqslant j_r \Rightarrow q_{i_r} \geqslant q_{j_r}$. Construct the sets $F_i(r, \alpha)$ as in (12). Because no confusion will arise, we will drop the subscript r and (r, α). (10) implies that

$$\forall i \in N, \forall w \in F_i, \qquad a(w) \geqslant q_i \cdot \pi(w; a^*). \qquad (13)$$

From (13), rearranging (11), using $q \geqslant 0$ and using Assumption 1 (the right hand side of (11) is nonnegative), it follows that $\sum_{i=1}^{n} (b_i - p_i) \cdot q_i > 0$. Let k be the first index for which $\sum_{i=1}^{k} (b_i - p_i) \cdot q_i > 0$. Such an index exists since $k = n$ satisfies the inequality. By definition of k, $(b_k - p_k) \cdot q_k > 0$. Since $q_k \geqslant 0$, it follows that $b_k - p_k > 0$. By construction, $q_{k-1} \geqslant q_k$. Hence, $\sum_{i=1}^{k-2} (b_i - p_i) \cdot q_i + (b_{k-1} - p_{k-1} + b_k - p_k) \cdot q_{k-1} > 0$. If the coefficient of q_{k-1} is nonpositive, the inequality implies that $\sum_{i=1}^{k-2} (b_i - p_i) \cdot q_i > 0$, but this is a contradiction since this implies that there is an index smaller than k than satisfies the inequality. Hence the coefficient of q_{k-1} is positive. Continuing recursively, we obtain the result since we show that $q_1 \cdot \sum_{i=1}^{k} (b_i - p_i) > 0$ and $q_1 > 0$. ∎

The reader will note that if an index k satisfies the condition in Proposition 7, then k is strictly less than n since $\sum_{i \leqslant n} (b_i - p_i(r, \alpha)) = 0$ for any permutation r and any vector of strategies α. We now show that this condition is also sufficient for nonexistence of neutral transfer rules when the excess liability is small and when β attains a maximum.

PROPOSITION 8. *Suppose that β attains a maximum and let $0<\beta^* \equiv \max_\alpha \beta(\alpha)$.*[5] *Suppose that B is such that $\bar{B}=n\cdot\beta^*$ and that there exist $\alpha^* \in \operatorname{argmax}\{\beta(\alpha); \alpha \in M\}$, a permutation r, and an index k such that $\sum_{i \leqslant k}(b_i - p_i(r, \alpha^*))>0$. Then, there does not exist a neutral and balanced transfer rule that is bounded with respect to B.*

Proof. Let α^*, k, and r satisfy the condition of the proposition. To simplify notation, we suppose that the order induced by r is $(1, 2, ..., n)$ and we write p_i instead of $p_i(r, \alpha^*)$. Note that by definition of α^*, and by the assumption on the total liability, the right hand side of (11) is equal to zero. Consider the family $\{F_i; i \in N\}$ that is induced by the permutation r. Let $q>0$ and define $\forall i \leqslant k$, $q_i = q$ and $\forall i \geqslant k+1$, $q_i = 0$. If $F_{k+1} = \varnothing$, choose $q>0$ arbitrary. If $F_{k+1} \neq \varnothing$, choose q such that

$$0<q \leqslant \min\left\{\frac{\pi(w; a^*\backslash\alpha_j^*) - \pi(w; a^*\backslash\alpha_i^*)}{\pi(w; a^*)} \,\middle|\, i \geqslant k+1, j \leqslant k, w \in F_i\right\}. \tag{14}$$

Such a choice is possible since by definition of the family $\{F_i \mid i \in N\}$ the right hand side of (14) is strictly positive. From (14), it follows that for any $i \in N$, any $w \in F_i$, $F_i \neq \varnothing$, and any $j \in N$,

$$q_i \cdot \pi(w; a^*) \geqslant q_j \cdot \pi(w; a^*) - (\pi(w; a^*\backslash\alpha_j^*) - \pi(w; a^*\backslash\alpha_i^*)). \tag{15}$$

Consider the function $a: \Omega \to \mathbb{R}$ defined by $\forall w \in F_i$, $a(w) \equiv q_i \cdot \pi(w; a^*)$. From (15), the function a satisfies (10). It follows that

$$\begin{aligned}\sum_{w \in \Omega} a(w) &= \sum_{i \in N} q_i \cdot \sum_{w \in F_i} \pi(w; a^*)\\ &= q \cdot \sum_{i \leqslant k} p_i.\end{aligned}$$

[5] The fact that $\beta^* = \max \beta(\alpha)$ is not trivial. Indeed, β is not in general a continuous function of α, hence a maximum may fail to exist. Suppose that $0<\beta^*<+\infty$. Then it is possible to show that a maximum exists if, and only if, there exists a sequence $\{\alpha^k\}$ such that $\beta(\alpha^k) \to \beta^*$ and such that $\rho(\alpha)>0$, where $\alpha = \lim \alpha^k$ (such a limit exists-by using a subsequence if necessary-by compactness of M). We observe that if $\beta^*=0$, then $t=0$ is a solution to the partnership problem.

Now, the left hand side of (11) can be rewritten as

$$\sum_{i\in N} B_i\cdot\Big(q_i-\sum_{w\in\Omega} a(w)\Big)=\sum_{i\in N} B_i\cdot\Big(q_i-q\cdot\sum_{j\leqslant k} p_j\Big)$$
$$=q\cdot\Big(\sum_{i\in N} B_i\Big)\cdot\Big(\sum_{j\leqslant k}(b_j-p_j)\Big)$$
$$>0,$$

where the inequality follows by assumption. But then, (10) and (11) hold, and Proposition 8 follows from Lemma 6. ▌

If the condition of Proposition 8 is satisfied, there does not exist a neutral solution whenever the total liability of the partnership is in a certain neighborhood of $n\cdot\beta^*$. Hence, even if there is excess liability in the partnership, it might be impossible to find neutral solutions. Proposition 8 points out the importance of the repartition of the total liability among the partners when the total liability is close to $n\cdot\beta^*$. Whenever there exist $i\in N$ and $a_i\in A_i$ for which $U_i(a^*\backslash a_i)>U_i(a^*)$, it is possible to find repartitions of the total liability that violate the condition of the proposition, even if there exists balanced solutions to the partnership problem. For instance, let α^* be chosen as in Proposition 8, and let i be such that $|\Omega(i;\alpha^*)|$ is minimum over N and hence strictly less than 1. (Indeed, observe that if $U_i(a^*\backslash a_i)-U_i(a^*)>0$ for each some i and a_i, obligatorily $\beta^*>0$. Consequently, there must exist a player i for which $|\Omega(i;\alpha^*)|<1$ for otherwise $\beta^*=+\infty$.) Any repartition of the total liability such that $b_i>|\Omega(i;\alpha^*)|$ will imply the nonexistence of a neutral solution. This observation leads to the following result.

COROLLARY 9. *Suppose that β has a maximum and that α^* is such that $\beta(\alpha^*)=\beta^*$. If the family $\{\Omega(i;\alpha^*)\}$ forms a partition of Ω, then there exists a neutral solution compatible with limited liability only if for each $i\in N$, $b_i=\sum_{w\in\Omega(i;\alpha^*)}\pi(w,a^*)$.*

Proof. Under the assumption of the corollary, for any permutation r, for any partner i, $F_{r(i)}(r,\alpha^*)=\Omega(i;\alpha^*)$, i.e., for any permutation r, $p_i(r,\alpha^*)=\sum_{w\in\Omega(i;\alpha^*)}\pi(w,a^*)$. From Proposition 8, there exists a neutral solution only if for each $i\in N$, $b_i\leqslant p_i(r,\alpha^*)$. Since $\sum_i p_i(r,\alpha^*)=\sum_i b_i=1$, the result follows. ▌

Corollary 9 has a strong implication. If for the worst collective deviation α^*, for any output w, there is a unique partner that minimizes $\pi(w;a^*\backslash\alpha_i^*)$, then there is a unique partition of the total liability that is compatible with the existence of a neutral solution. Moreover, the relative liability of each partner i must be equal to the $\pi[a^*]$-measure of the set of states whose probability of occurrence is minimum when i deviates to α_i^*. The example of Section 8 is an example of such a case.

7. Individual Rationality

If the partners have outside opportunities, the partnership will form only if the expected utility of each partner is larger than her reservation utility, i.e., the level of utility corresponding to her next best opportunity. Let $\underline{U}_i$ be partner i's reservation utility level. A necessary condition for the partnership to form is that

$$\sum_{i \in N} [U_i(a^*) - \underline{U}_i] \geqslant 0. \tag{16}$$

We can think of $U_i(a^*) - \underline{U}_i$ as the *first best opportunity cost* for partner i of participating in the partnership. Note that by risk neutrality, the sum in (16) is the sum of the opportunity costs not only in the first best (i.e., without transfers) but also in a solution involving side-payments. To avoid trivialities, we will suppose that (16) holds. The sharing rule t satisfies individual rationality if $EU_i(a^*; t) \geqslant \underline{U}_i$ for each partner i, i.e., if

$$\forall i \in N, \qquad \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a^*) \geqslant \underline{U}_i - U_i(a^*). \tag{17}$$

The existence of individual rational sharing rules that sustain efficiency is immediate if the condition of Theorem 2 holds. Indeed, consider any solution t and define a new sharing rule $\tilde{t}$ such that

$$\forall i \in N, \forall w \in \Omega, \qquad \tilde{t}_i(w) = t_i(w) - \sum_{\tilde{w} \in \Omega} t_i(\tilde{w}) \cdot \pi(\tilde{w}; a^*) + \underline{U}_i - U_i(a^*) + \frac{1}{n} \cdot \sum_{i \in N} [U_i(a^*) - \underline{U}_i].$$

Then, from (16)

$$\sum_{w \in \Omega} \tilde{t}_i(w) \cdot \pi(w; a^*) = \underline{U}_i - U_i(a^*) + \frac{1}{n} \cdot \sum_{i \in N} [U_i(a^*) - \underline{U}_i] \geqslant \underline{U}_i - U_i(a^*),$$

and for each $w \in \Omega$,

$$\sum_{i \in N} \tilde{t}_i(w) = 0.$$

Hence, $\tilde{t}$ is an individually rational transfer rule that implements efficiency.

Individual rationality can be difficult to satisfy when imposed in conjunction with limited liability. There exist necessary and sufficient conditions for the existence of an individually rational transfer rule that satisfies limited liability and that sustains efficiency. These conditions are very similar to the conditions of Lemma 6: (10) is unchanged, and the condition corresponding to (11) is

$$\sum_{i \in N} \left\{ q_i \cdot [B_i + \underline{U}_i - U_i(a^*)] - B_i \cdot \sum_{w \in \Omega} a(w) \right\} > \left[\sum_{i \in N} B_i - n \cdot \beta(\alpha) \right] \cdot \rho(\alpha). \tag{18}$$

It follows that if, for each partner i, $\underline{U}_i \leqslant U_i(a^*)$, the existence of a neutral transfer rule implies the existence of an individually rational transfer rule (the reverse is not true in general).

Proposition 10 (resp. 11) gives a necessary (resp. sufficient) condition for the nonexistence of individually rational transfer rules when limited liability is imposed. Most of the notation is borrowed from Section 6. We define

$$\forall i \in N, \qquad \delta_i = \frac{U_i(a^*) - \underline{U}_i}{\bar{B}}. \tag{19}$$

δ_i is positive if, and only if, partner i's first best opportunity cost is positive. The reader will note that (16) is compatible with $\delta_i < 0$ for some partner i.

PROPOSITION 10. *Suppose that Assumption* 1 *holds. If there does not exist an individual rational and balanced transfer rule that sustains a^* and is bounded with respect to B, where B is some partition of the total liability $\bar{B}$, then there exist a vector of mixed strategies α, a permutation $r \in R$, and an index $k < n$ such that*

$$\sum_{i \leqslant k} (b_i - \delta_i - p_i(r, \alpha)) > 0.$$

PROPOSITION 11. *Suppose that β attains a maximum and let $0 < \beta^* \equiv \max_\alpha \beta(\alpha)$. Suppose that B is such that $\bar{B} = n \cdot \beta^*$ and that there exist $\alpha^* \in \operatorname{argmax}\{\beta(\alpha); \alpha \in M\}$, a permutation r, and an index k such that $\sum_{i \leqslant k} (b_i - \delta_i - p_i(r, \alpha^*)) > 0$. Then there does not exist an individual rational and balanced transfer rule that is bounded with respect to B.*

Proofs. Similar to the proofs of Propositions 7 and 8. ∎

Hence, with individual rationality, what matters is not only the relative liability of partner i but also the first best opportunity cost of participating in the partnership. This tradeoff is illustrated in the example of Section 8.

8. An Example

Let $N=2$, $\Omega=\{w_1, w_2\}$, $\forall i=1,2$, $A_i=\{0,1\}$, $a^*=(1,1)$. Suppose that $U_1(a^*\backslash 0)-U_1(a^*)=-1$, $U_2(a^*\backslash 0)-U_2(a^*)=1/2$, $\pi[a^*]=\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]$, and $\pi[a^*\backslash 0]=\left[\begin{smallmatrix}1/4\\3/4\end{smallmatrix}\right]$. Then $\beta(\alpha)$ is maximized for $\alpha^*=(1,0)$ and $\beta^*=1$. From Theorem 5, there exists a balanced transfer rule that sustains a^* and that satisfies limited liability if, and only if, $\bar{B}\geqslant 2$. The incentive compatibility constraints are

$$t_1(w_1)-t_1(w_2)\geqslant -4 \tag{IC1}$$

$$t_2(w_1)-t_2(w_2)\geqslant 2. \tag{IC2}$$

By choosing t such that $t_1(w_1) = -2\cdot s_1, t_1(w_2) = 2\cdot s_2, t_2(w_1) = 2\cdot s_1, t_2(w_2)=-2\cdot s_2$, the two incentive compatibility conditions are satisfied and for each $i=1,2$ and each $w\in\Omega$, $t_i(w)\geqslant -B_i$, where $B_i=2\cdot s_i$. Hence, whenever the total liability is greater than 2, for any repartition of this liability between the two partners there exists a transfer rule that will sustain α^* and that is bounded with respect to B.

Here, $\Omega(1;\alpha^*)=\{w_2\}$ and $\Omega(2;\alpha^*)=\{w_1\}$. From Proposition 8, there exists a neutral solution if, and only if, $s_1=s_2=1/2$. Indeed, $p(1;r,\alpha^*)=p(2;r,\alpha^*)=1/2$ for any permutation r. Hence, $s_1\leqslant 1/2$ and $s_2\leqslant 1/2$. Since $s_1+s_2=1$, the result follows. A direct proof is possible in this example. Neutrality implies that for each $i=1,2$, $t_i(w_1)=-t_i(w_2)$. Because t is balanced, there exists k such that $-t_1(w_1)=t_1(w_2)=t_2(w_1)=-t_2(w_2)=k$. But then (IC1) and (IC2) imply that $k\in[1,2]$, or $-k\in[-2,-1]$. Limited liability imposes that $\forall i\in N$, $-k\geqslant -2\cdot s_i$. If $s_i<1/2$ for some i, then $-2\cdot s_i>-1$, and this contradicts incentive compatibility.

The existence of an individual rational transfer rule that is compatible with limited liability depends on the values of $\underline{U}_1-U_1(a^*)$ and $\underline{U}_2-U_2(a^*)$ as well as on the relative liability levels s_1 and s_2. Applying Proposition 11, there exists an individual transfer rule compatible with limited liability if, and only if, for each partner i, $s_i-\delta_i\leqslant\frac{1}{2}$, which implies that $s_i\leqslant\frac{1}{2}+\delta_i$. We know that $\delta_1+\delta_2\geqslant 0$ if the partnership forms. If $\delta_1>0$ and $\delta_2<0$, then it is necessary that partner 1 have a higher relative liability than partner 2; this is because partner 2 must be compensated for belonging to the partnership by a positive expected transfer if individual rationality is to be satisfied.

9. Conclusion

We characterize the environments for which a partnership can sustain efficiency through monetary transfers even if no information can be gained by observing the realization of the output. We introduce a concept of distance between deviations and we show that the possibility to sustain a particular vector of actions is related to the distance of deviations with respect to the social gain to the partnership when the partners deviate. This leads to the definition of an index β of the likelihood of deviations in a partnership. We show that it is possible to sustain a particular vector of actions if, and only if, the index is bounded by a finite positive number.

If the partners have limited liability, i.e., if the transfers are bounded below by a finite vector, then we show that it is possible to sustain a vector of actions by transfers that satisfy a no-bankruptcy condition if, and only if, the average liability in the partnership is greater than the supremum of the index β over all possible deviations. Thus, the existence of a transfer rule consistent with bankruptcy constraints depends only on the total liability in the partnership and the number of partners: the partition of the total liability among the partners does not affect existence. However, the partition of the total liability plays a role for the existence of neutral transfer rules and of individually rational transfer rules.

Some results of this paper depend crucially on the particular form of the utility functions. For instance, the result that the existence of budget balanced transfer rules satisfying limited liability and implementing the first best depends only on the level of the average liability is linked to the assumption of quasi-linearity. While the assumption of risk neutrality seems to be crucial for most of the results of this paper, separability does not seem to play such an important role. Further work is needed in order to extend our results to general utility functions and to understand the exact role that risk neutrality and separability play for the implementation problem.

Throughout the paper we have confined our attention to the sustainability of pure strategy profiles. This is not so restrictive in our model because it is always possible to find a pure strategy profile which is efficient. Nevertheless, the extension of our work to the mixed strategy case could be interesting from the following point of view. Legros [10] showed that in the deterministic case, ε-efficiency can be sustained in mixed strategies even if our necessary and sufficient condition is violated, e.g., when the partners are symmetric. It is reasonable to conjecture that with stochastic output ε-efficiency can also be sustained in mixed strategies. One of us is currently completing a paper proving this conjecture.

Fudenberg *et al.* [6] analyze a repeated partnership model with imperfect monitoring and give a sufficient condition for ε-sustainability which is called pairwise identifiability. Pairwise identifiability at the efficient vector of actions a^* is actually stronger than our necessary and sufficient condition in Theorem 2. It will be interesting to consider ε-sustainability of efficiency in mixed strategies based on the ideas addressed in this paper instead of the idea of pairwise identifiability, because of the possibility of sharpening Fudenberg *et al*'s [6] results.

APPENDIX

Proof of Lemma 1

(ii) is the generalization of the condition that the distance between two elements is equal to zero if, and only if, these elements coincide. (iii) is the generalization of the triangle inequality. Observe that by construction of d, $\tilde{d}$ satisfies the symmetry condition for a metric. Hence (iv) follows from (i)–(iii). (i) is immediate. For (ii), sufficiency is obvious. For necessity, let $Q \in \Delta(\Omega)$ and suppose that $d(Q)=0$ and that there exists two distinct elements q and $\hat{q}$ in Q. Since Q is included in a simplex, it must be true that there exists $w \in \Omega$ for which $q(w) > \hat{q}(w)$. But then,

$$\sum_{w \in \Omega} q(w) > \sum_{w \in \Omega} \inf_{\tilde{q} \in Q} \tilde{q}(w).$$

But this contradicts the fact that $d(Q)=0$ and $q \in \Delta(\Omega)$. To prove (iii), let Q, T, V be three subsets of $\Delta(\Omega)$. By definition of d,

$$d(Q \cup T) + d(T \cup V) = 2 - \sum_{w \in \Omega} \left(\inf_{q \in S \cup T} q(w) + \inf_{q \in T \cup V} q(w) \right). \tag{a}$$

We observe that for any $w \in \Omega$,

$$\begin{aligned}
&\inf_{q \in S \cup T} q(w) + \inf_{q \in T \cup V} q(w) \\
&\quad = \min \Big\{ \inf_{q \in S} q(w) + \inf_{q \in T} q(w),\ \inf_{q \in S} q(w) + \inf_{q \in V} q(w), \\
&\qquad\quad \inf_{q \in T} q(w) + \inf_{q \in V} q(w),\ 2 \cdot \inf_{q \in T} q(w) \Big\} \\
&\quad \leqslant \min \Big\{ \inf_{q \in S} q(w) + \inf_{q \in T} q(w),\ \inf_{q \in T} q(w) + \inf_{q \in V} q(w) \Big\} \\
&\quad = \min \Big\{ \inf_{q \in S} q(w),\ \inf_{q \in V} q(w) \Big\} + \inf_{q \in T} q(w) \\
&\quad \leqslant \min \Big\{ \inf_{q \in S} q(w),\ \inf_{q \in V} q(w) \Big\} + q^0(w) \\
&\quad = \inf_{q \in S \cup V} q(w) + q^0(w),
\end{aligned}$$

where $q^0 \in T$ is an arbitrary element of T. By summing over $w \in \Omega$, and substituting in (a), the result follows. (v) is a monotonicity property of d and its proof follows the property of the inf operator.

Proof of Theorem 2

Recall that $\Omega=\{w_1, ..., w_l\}$. We rewrite the partnership problem into a linear algebraic problem. By definition of a probability measure,

$$\forall a \in A, \qquad \pi(w_l; a) = 1 - \sum_{w \neq w_l} \pi(w; a).$$

As a consequence, the incentive compatibility conditions can be rewritten

$\forall i, \forall a_i \in A_i,$

$$\sum_{w \neq w_l} [\pi(w; a^*) - \pi(w; a^* \backslash a_i)] \cdot [t_i(w) - t_i(w_l)] \geqslant U_i(a^* \backslash a_i) - U_i(a^*).$$

By using the budget balance condition, the incentive constraints for partner n become

$$\forall a_n \in A_n, \qquad -\sum_{w \neq w_l} [\pi(w; a^*) - \pi(w; a^* \backslash a_n)] \cdot \sum_{i \neq n} [t_i(w) - t_i(w_l)]$$
$$\geqslant U_n(a^* \backslash a_n) - U_n(a^*).$$

For each partner i, let P_i be the $T_i \times l$ matrix whose (j, w)th element is equal to

$$P_i(j, w) = \begin{cases} \pi(w; a^*) - \pi(w; a^* \backslash a_i(j)) & \text{if} \quad w \neq w_l \\ \sum_{w \neq w_l} (\pi(w; a^* \backslash a_i(j)) - \pi(w; a^*)) & \text{if} \quad w = w_l. \end{cases}$$

Let P be the $\sum_{i \in N} T_i \times (n-1) \cdot l$ matrix

$$P = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{n-1} \\ -P_n & -P_n & \cdots & -P_n \end{bmatrix}.$$

Let $\mathbf{t}_i$ be the column vector of dimension l whose jth element is $t_i(w_j)$ and let $\mathbf{t}$ be the $(n-1) \cdot l$-dimensional column vector obtained by superposing the vectors $\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_{n-1}$, in this order. Finally, let $\mathbf{u}_i$ be the T_i-dimensional column vector whose j-th element is $\mathbf{u}_i(j) = U_i(a^* \backslash a_i(j)) - U_i(a^*)$ and let $\mathbf{u}$ be the column vector with $\sum_{i \in N} T_i$ rows obtained by superposing the vectors $\mathbf{u}_i$ in the order defined by $1, 2, ..., n$.

It is immediate that the partnership's problem has a solution if and only if the system $P \cdot \mathbf{t} \geqslant \mathbf{u}$ is consistent. We will use the following result of Fan [4, Theorem 1).

LEMMA A. *The system $P \cdot \mathbf{t} \geqslant \mathbf{u}$ has a solution whose norm is bounded if, and only if, whenever there exists $\lambda \in \mathbb{R}_+^{\sum_i T_i}$ such that $\lambda \cdot P = 0$ then $\lambda \cdot \mathbf{u} \leqslant 0$.*

For a given vector of mixed strategies α, we will use the following notation:

$$\mathbf{u}_i(\alpha_i) \equiv \sum_{a_i \in A_i} \alpha_i(a_i) \cdot \mathbf{u}_i(a_i)$$

$$P_i(w, \alpha_i) \equiv \sum_{a_i \in A_i} \alpha_i(a_i) \cdot P_i(w, a_i)$$

$$P_i(\alpha_i) \equiv (P_i(w_1, \alpha_i), ..., P_i(w_l, \alpha_i)).$$

Condition C. $\forall \alpha \in M, \forall z \in \mathbb{R}_+^n$, *either* $\exists i < n$, *such that* $z_i \cdot P_i(\alpha_i) \neq z_n \cdot P_n(\alpha_n)$, *or* $\sum_i z_i \cdot \mathbf{u}_i(\alpha_i) \leqslant 0$.

LEMMA B. *The partnership problem has a solution if, and only if, Condition* C *holds.*

Proof of Lemma B. This is a restatement of Lemma A. For $\alpha \in M$, $z \in \mathbb{R}_+^n$ given, let us define a vector λ such that $\forall a_i \in A_i$, $\lambda(a_i) = z_i \cdot \alpha_i(a_i)$. Then Condition C implies the condition of Lemma A since by varying α and z one can obtain all the vectors λ in $\mathbb{R}_+^{\sum_i T_i}$. Reciprocally, if the condition of Lemma A holds, then for any λ in $\mathbb{R}_+^{\sum_i T_i}$, it is possible to define the nonnegative real numbers z_i such that $\forall i \in N$, $z_i \equiv \sum_{a_i \in A_i} \lambda(a_i)$. If $z_i > 0$, we define $\alpha_i(a_i) \equiv \lambda(a_i)/z_i$. Otherwise, we choose any mixed strategy in M_i. This leads to Condition C. ∎

Lemma B is also true if the weak inequality is replaced by a strict inequality when $\sum_i z_i \cdot \mathbf{u}_i(\alpha_i) < 0$ appears in Condition C (Fan [4, Theorem 6].) In this case, if the new Condition C is satisfied, a^* is a strict Nash equilibrium (i.e., being efficient is the only best response if the other partners are efficient).

To prove Theorem 2, it is enough to prove that Condition C is equivalent to the condition that $\beta^* < +\infty$, where $\beta^* \equiv \sup\{\beta(\alpha) | \alpha \in M\}$.

(Show $C \Rightarrow \beta^* < +\infty$) Suppose that Condition C holds. Consider $\alpha \in M$. Suppose that $\sum_i \mathbf{u}_i(\alpha_i) > 0$ (otherwise, $\beta(\alpha) \leqslant 0$). Then, setting $\forall i \in N$, $z_i = 1$ in Condition C, there must exist an index i such that $P_i(\alpha_i) \neq P_n(\alpha_n)$, i.e., $\pi[a^* \backslash \alpha_i] \neq \pi[a^* \backslash \alpha_n]$. But then $\rho(\alpha) > 0$ which proves that $\beta(\alpha) < +\infty$ since $\sum_i \mathbf{u}_i(\alpha_i) < +\infty$. Suppose now that $\beta^* = +\infty$. By definition of the supremum, there exists a sequence $\{\alpha^k\}$ such that $\beta(\alpha^k) \to +\infty$. Since M

is compact, there exists α^* such that $\alpha^k \to \alpha^*$ (in the product topology on M). $\beta(\alpha^k) \to +\infty$ only if $\rho(\alpha^k) \to 0$ and if $\sum_{i \in N} \mathbf{u}_i(\alpha_i^k) \to \delta$, $\delta > 0$. By continuity of $\sum_{i \in N} (U_i(a^* \backslash \hat{\alpha}_i) - U_i(a^*))$ and of $\rho(\hat{\alpha})$ in $\hat{\alpha}$, it follows that $\beta(\alpha^*) = +\infty$, which is a contradiction since $\alpha^* \in M$.

(Show $\beta^* < +\infty \Rightarrow$C) Suppose that $\beta^* < +\infty$ but that Condition C is violated. Then, there exist a vector of mixed strategies α and a non-negative vector z such that

$$\forall i, \qquad z_i \cdot P_i(\alpha_i) = z_n \cdot P_n(\alpha_n) \qquad \text{and} \tag{a.1}$$

$$\sum_i z_i \cdot \mathbf{u}_i(\alpha_i) > 0. \tag{a.2}$$

(a.2) implies that there exists i such that $z_i > 0$ and $\mathbf{u}_i(\alpha_i) > 0$, i.e.,

$$U_i(a^* \backslash \alpha_i) > U_i(a^*). \tag{a.3}$$

Case 1. $z_n \cdot P_n(\alpha_n) = 0$. Since $z_i > 0$, (a.1) implies that $P_i(\alpha_i) = 0$. This implies that $\pi[a^*] = \pi[a^* \backslash \alpha_i]$. Consider a new strategy $\hat{\alpha}$ such that $\hat{\alpha}_i = \alpha_i$ and for any $j \neq i$, $\hat{\alpha}_j(a_j^*) = 1$. Hence, $\pi[a^* \backslash \hat{\alpha}_i] = \pi[a^* \backslash \hat{\alpha}_j]$. Because $\mathbf{u}_i(\hat{\alpha}_i) > 0$ and $\forall j \neq i$, $\mathbf{u}_j(\hat{\alpha}_j) = 0$, $\beta(\hat{\alpha}) = +\infty$, which contradicts our assumption.

Case 2. $z_n \cdot P_n(\alpha_n) \neq 0$. Since $z \geqslant 0$, it follows from (a.1) that $\forall j$, $z_j > 0$. Without loss of generality, we suppose that n is such that $\forall j$, $z_n \geqslant z_j$. Let $x_j \equiv z_j / z_n$, $x_j \in (0, 1]$ by construction. (a.1) can be rewritten as

$$(1 - x_j) \cdot \pi[a^*] + x_j \cdot \pi[a^* \backslash a_j] = \pi[a^* \backslash \alpha_n]. \tag{a.4}$$

Let $\alpha_j[x_j] \equiv (1 - x_j) \cdot a_j^* + x_j \cdot \alpha_j$ be a new mixed strategy for player $j \neq n$. $\alpha[x_j]$ is a mixture of α_j and of the pure strategy a_j^*. Let $\alpha[x]$ be the resulting vector of mixed strategies. (a.1) implies that

$$\forall j, \qquad \pi[a^* \backslash \alpha_j[x_j]] = \pi[a^* \backslash \alpha_n[x_n]].$$

Hence, $\rho(\alpha[x]) = 0$. Observe that

$$\begin{aligned} \sum_j z_j \cdot \mathbf{u}_j(\alpha_j) &= z_n \cdot \sum_j x_j \cdot \mathbf{u}_j(\alpha_j) \\ &= z_n \cdot \sum_j \mathbf{u}_j(\alpha_j[x_j]). \end{aligned}$$

Hence, $\sum_j \mathbf{u}_j(\alpha_j[x_j]) > 0$ since $z_n > 0$. This implies that $\beta(\alpha[x]) = +\infty$ which is a contradiction.

Proof of Eqs. (2) *and* (3)

Let I be the $l \times l$ identity matrix. Define $\hat{P}_i \equiv \left[\begin{smallmatrix} P_i \\ I \end{smallmatrix}\right]$ and let $\hat{P}$ be defined by

$$\hat{P} = \begin{bmatrix} \hat{P}_1 & 0 & \cdots & 0 \\ 0 & \hat{P}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{P}_{n-1} \\ -\hat{P}_n & -\hat{P}_n & \cdots & -\hat{P}_n \end{bmatrix}.$$

Let $\mathbf{b}_i = (-B_i, ..., -B_i)$ be the l-dimensional column vector in which each component is equal to $-B_i$ and let $\hat{u}_i \equiv \left[\begin{smallmatrix} \mathbf{u} \\ \mathbf{b}_i \end{smallmatrix}\right]$. $\hat{u}$ denotes the column vector obtained by superposing the vectors $\hat{u}_i$, $i = 1, 2, ..., n$, in this order. Then, there exists a balanced transfer rule that is bounded with respect to B if and only if the system $\hat{P} \cdot \mathbf{t} \geqslant \hat{u}$ has a solution. Applying Lemma A to this system, the *nonexistence* of a solution to such a system is equivalent to the existence of functions $d_i : \Omega \to \mathbb{R}_+$, $k : \Omega \to \mathbb{R}$ such that the following two conditions hold:

$$\sum_{i \in N} (U_i(a^* \backslash \alpha_i) - U_i(a^*)) > \sum_{i \in N} B_i \cdot \sum_{w \in \Omega} d_i(w) \tag{a.5}$$

$$\pi(w; a^* \backslash \alpha_i) - \pi(w; a^*) - d_i(w) = k(w). \tag{a.6}$$

We show that this is equivalent to the opposite of the condition of Theorem 5. Let $k' : \Omega \to \mathbb{R}$ be defined by $\forall w \in \Omega$, $k'(w) = k(w) + \pi(w; a^*)$. From (a.6), for each i and each w, $\pi(w; a^* \backslash \alpha_i) - k'(w) = d_i(w)$. Hence,

$$\forall i \in N, \qquad \sum_{w \in \Omega} d_i(w) = 1 - \sum_{w \in \Omega} k'(w) \tag{a.7}$$

$$\forall w \in \Omega, \qquad k'(w) \leqslant \min_{i \in N} \pi(w; a^* \backslash \alpha_i), \tag{a.8}$$

where (a.8) uses the fact that $d_i \geqslant 0$ for all $i \in N$. Using (a.5)–(a.8), it follows that

$$\sum_{i \in N} (U_i(a^* \backslash \alpha_i) - U_i(a^*)) > \sum_{i \in N} B_i \cdot \rho(\alpha). \tag{a.9}$$

Clearly, if $\rho(\alpha) \neq 0$, this is equivalent to $\sum_{i \in N} B_i < n \cdot \beta(\alpha)$. If $\rho(\alpha) = 0$, the left hand side of (a.9) is positive. Hence, $\beta(\alpha) = +\infty$ and $\sum_{i \in N} B_i < n \cdot \beta(\alpha)$. Suppose that $\rho(\alpha) = 0$ and that $\sum_{i \in N} B_i < n \cdot \beta(\alpha)$. By definition of β and since $B \geqslant 0$, it must be true that $\sum_{i \in N} (U_i(a^* \backslash \alpha_i) - U_i(a^*)) > 0$, for if $\sum_{i \in N} (U_i(a^* \backslash \alpha_i) - U_i(a^*)) \leqslant 0$, $\beta(\alpha) \leqslant 0$ which contradicts the assumption. This proves Theorem 5. ∎

Proof of Lemma 6

We observe that there exists a balanced transfer rule that is neutral if, and only if, there exists a balanced transfer rule that satisfies

$$\forall i \in N, \qquad \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a^*) \geqslant 0. \tag{a.10}$$

Clearly, (a.10) must hold if t is a neutral transfer rule. Suppose that (a.10) holds and that t is a balanced transfer rule. By using the balance condition, we have

$$\forall j \in N, \qquad -\sum_{i \in N \setminus \{j\}} \sum_{w \in \Omega} t_i(w) \cdot \pi(w; a^*) \geqslant 0.$$

But the left hand side is nonpositive. Hence, it must be true that (a.10) holds with an equality.

Let $e \in \mathbb{R}^l$ be a row vector whose w_kth element is equal to $\pi(w_k; a^*)$. Let $\tilde{P}_i$ be the matrix obtained by superposing the matrix $\hat{P}_i$ constructed previously and the row vector e, i.e., $\tilde{P}_i = [{}^{\hat{P}_i}_{e}]$. Let $\tilde{P}$ be the resulting matrix

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 & \cdots & 0 \\ 0 & \tilde{P}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{P}_{n-1} \\ -\tilde{P}_n & -\tilde{P}_n & \cdots & -\tilde{P}_n \end{bmatrix}.$$

Let $\tilde{u}_i$ be the column vector obtained by adding the element 0 to the vector $\hat{u}_i$ that we constructed above, and let $\tilde{u}$ be the vector obtained by superposing the n vectors $\tilde{u}_i$. From our previous reasoning, the system $\tilde{P} \cdot \mathbf{t} \geqslant \tilde{u}$ incorporates the conditions that t is a balanced transfer rule that is bounded with respect to B and that is neutral. By using the same reasoning as before, it is possible to show that there *does not* exists a solution to this new system if, and only if, there exist maps $d_i: \Omega \to \mathbb{R}_+$, $k: \Omega \to \mathbb{R}$, and a nonnegative vector $q \in \mathbb{R}^n_+$ such that

$$\pi(w; a^* \backslash \alpha_i) - d_i(w) - q_i \cdot \pi(w; a^*) = k(w), \qquad \text{all } i \in N, \quad \text{all } w \in \Omega \tag{a11}$$

$$\sum_{i \in N} (U_i(a^* \backslash \alpha_i) - U_i(a^*)) > \sum_{i \in N} B_i \cdot \sum_{w \in \Omega} d_i(w). \tag{a.12}$$

We want to show that (a.11) and (a.12) are equivalent to (10), (11). Suppose that (a.11) and (a.12) hold. Let $\forall w \in \Omega$, $a(w) \equiv \min_{i \in N} \pi(w; a^* \backslash \alpha_i) - k(w)$. Then, from (a.11), $\forall i \in N$, $\forall w \in \Omega$,

$$d_i(w) = \pi(w; a^* \backslash \alpha_i) - \min_{j \in N} \pi(w; a^* \backslash \alpha_j) + a(w) - q_i \cdot \pi(w; a^*). \tag{a.13}$$

Because $d_i \geqslant 0$, (10) must hold. By summing (a.13) over $w \in \Omega$, we obtain $\sum_{w \in \Omega} d_i(w) = \rho(\alpha) - q_i + \sum_{w \in \Omega} a(w)$. Algebraic manipulations show that (a.12) can be rewritten as (11).

Suppose now that (10) and (11) hold. Define

$$d_i(w) \equiv \pi(w; a^* \backslash \alpha_i) - \min_{j \in N} \pi(w; a^* \backslash \alpha_j) + a(w) - q_i \cdot \pi(w; a^*)$$
$$k(w) \equiv \min_{j \in N} \pi(w; a^* \backslash \alpha_j) - a(w).$$

Observe that (10) implies that $d_i(w) \geqslant 0$ for all i and for all w. Then, $\forall i \in N$, $\forall w \in \Omega$, $\pi(w; a^* \backslash \alpha_i) - d_i(w) - q_i \cdot \pi(w; a^*) = k(w)$, i.e., (a.11) holds. (11) implies (a.12) after some simple algebraic manipulations. This proves Lemma 6.

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