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# Allocating Joint Costs by Means of the Nucleolus[1]

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*Abstract:* This paper presents a sufficient condition for the nucleolus to coincide with the SCRB method vector and for nonemptiness of the core. It also studies the reasonableness and the monotonicity of the nucleolus under this condition. Finally it analyses the class of games satisfying the condition and compares it with the classes of convex games, subconvex games and the class $\tilde{Q}$ of Driessen and Tijs.

## 1 Introduction

Regulatory authorities such as the Federal Communications Commision or the Tennessee Valley Authority historically have determined tariffs based on the principle of fully distributed costs. This type of pricing has been analysed in Ransmeier (1942), Heaney and Dickinson (1982), Heaney (1979), James and Lee (1971), Young et al. (1980) and Braeutigam (1980).

Shubik (1962) made one of the first applications of game theory to the study of the joint costs allocation problem. Subsequently many authors (Littlechild 1975; Loehman and Whinston 1974; Champsaur 1975; Sharkey 1982a) have established the methodological similarity between this problem and the problem of finding a solution in an $n$-person cooperative game. Specifically, consider an enterprise producing a certain service for a set of $N$ markets. Associate with each subset $S$ of $N$ the cost $C(S)$ of producing the service for the $|S|$ markets alone. Faulhaber (1975) shows that if the regulator imposes a zero profit constraint, the existence of subsidy-free prices is equivalent to the nonemptiness of the core of the game $\langle N, C\rangle$. Sufficient conditions for the existence of the core can be found in Sharkey and Telser (1978).

The sufficient condition presented below defines a class of games with nonempty cores for which the Separable Cost Remaining Benefit (SCRB) method coincides with the nucleolus of the cost game. The coincidence of the SCRB method and the nucleolus

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has already been observed for a 3-person game in Straffin and Heaney (1981) and for a 5-person game in Suzuki and Nakayama (1976). Suzuki and Nakayama (1974) also obtain this technical result in the general case for a weaker condition than is stated here.[3] One of the purpose of the present paper is to show that the stronger condition provides an intuitive motivation for behavior which leads to this result. Independently, Driessen and Tijs (1983a) prove that the $\tau$-value of Tijs (1981) coincides with the nucleolus and with the SCRB vector for all games belonging to a certain class. We show that this class is disjoint from ours, except for the case of highly symmetrical games.

The paper is organized as follows. Section 2 presents a condition which defines a new class of games and for which the core is nonempty and the SCRB method leads to the nucleolus. Section 3 analyses the reasonableness and the monotonicity of the nucleolus for this class. Section 4 compares this class with those defined by Shapley (1971), Sharkey (1982b) and Driessen and Tijs (1983a). Section 5 presents some final comments.

## 2 The SCRB Method and the Nucleolus

Let $\langle N, v \rangle$ be the saving game associated with the cost allocation problem. The function $v$ is defined by,

$$v(S) = \sum_{i \in S} C(i) - C(S), \quad \text{for all } S \subset N$$

$$v(\phi) = 0.$$

Note that $v$ is always zero-normalized ($v(i) = 0$ for all $i$ in $N$). We require only that $v$ be monotonic,

$$v(S) \geqslant v(T), \quad \text{for all } T \subset S \subset N.$$

The set of imputations is defined as,

$$X = \{x \in \mathbb{R}^n \text{ s.t. } x(N) = v(N) \text{ and } x_i \geqslant 0,\ i = 1, \ldots, n\}$$

where $x(S)$ denotes $\sum_{i \in S} x_i$.

Clearly, every individually rational full cost allocation scheme $y$ (i.e. $y_i \leqslant C(i)$ and $y(N) = C(N)$) corresponds to only one vector $x$ in $X$; precisely, $y = -x + c$, where $c = (C(1), \ldots, C(n))$.

The SCRB method defines a vector in $\mathbb{R}^n$ such that every player $i$ gets his/her separable contribution ($SC_i$) and a fraction of the non-separable contribution ($NSC$). Formally, if $x$ is this vector,

$$x_i = SC_i + \alpha_i \cdot NSC, \quad i = 1, \ldots, n$$

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3 I thank a referee for pointing out this reference to me.

where

$$\alpha(N) = 1, \quad \alpha_i \geqslant 0, \quad i = 1, \ldots, n$$

$$SC_i = v(N) - v(N - i), \quad i = 1, \ldots, n$$

$$NSC = v(N) - SC(N)$$

It is immediate that $x$ is always Pareto optimal, but that $x_i$ may be positive or negative and that $x$ may not be an element of the core. We now consider the naive case where $\alpha_i = 1/n$ for all $i$ in $N$ and we denote by $\tilde{x}$ the vector $\tilde{x} = SC + NSC/n$.

The excess of a coalition $S$ with respect to (w.r.t.) $x$ is defined by $e(S, x) = v(S) - x(S)$. If the excesses are arranged in decreasing order, let $\Theta(x)$ be the resulting vector in $\mathbb{R}^{2^n - 1}$. An imputation $x$ is preferred to an imputation $y$ whenever $\Theta(x)$ is smaller than $\Theta(y)$ w.r.t. the lexicographical ordering on $\mathbb{R}^{2^n - 1}$. The nucleolus is the set of points in $X$ which are most preferred in this quasi-ordering on $\mathbb{R}^n$. Schmeidler (1969) proves that the nucleolus always exists, is unique, and is an element of every $\epsilon$-core. An intuitive interpretation is that the nucleolus minimizes dissatisfaction, with priority to coalitions which are most dissatisfied.

We give the following definition before stating a first result.

*Definition:* A game is pseudo-convex if $SC_i \geqslant v(S) - v(S - i)$, for all $i$ in $N$ and all $S$ in $N$.

Pseudo-convexity requires that the maximum of the separable contribution of a player to all coalitions is reached at the highest level of cooperation, i.e. $N$. Contrary to Shapley's notion of convexity, there is no snow ball effect here, but only an incentive for the players to reach the highest level of cooperation.

*Theorem 1:* If $v$ is pseudo-convex and satisfies the condition,

(PI) $n \cdot SC_i \geqslant -(n-1) \cdot NSC$

$\tilde{x} = SC + NSC/n$ is the nucleolus of the game and the core is nonempty.

*Proof:* Let (PCG) and (PCGI) denote respectively the class of pseudo-convex games and the class of pseudo-convex games which satisfy (PI). Let $S = \{i_1, \ldots, i_s\}$ be any coalition with $s$ members and denote $S_k$ the coalition obtained from $S$ by deleting the $k$ first players, $S_k = S - \{i_1, \ldots, i_k\}$ with $k = 1, \ldots, s$, and let $S_0 = S$. If $v$ is in (PCG),

$$SC_{i_k} \geqslant v(S_{k-1}) - v(S_k), \quad k = 1, \ldots, s.$$

Summing over the elements of $S - i_s$ and noting $v(i) = 0$ for all $i$ in $N$,

$$\sum_{k=1}^{s-1} SC_{i_k} \geqslant \sum_{k=1}^{s-1} (v(S_{k-1}) - v(S_k)) = v(S). \tag{1}$$

If we consider $S = N - j$, it is immediate that (1) implies,

$$NSC \leqslant 0. \tag{2}$$

From (PI) and (2), we have $\tilde{x}_i \geqslant 0$ for all $i$ in $N$. Since $\tilde{x}(N) = v(N)$ we have $\tilde{x} \in X$.

If $s \leqslant n-2$, by (1), (2) and (PI),

$$\begin{aligned} e(N-i, \tilde{x}) - e(S, \tilde{x}) &= NSC/n - (v(S) - SC(S) - s \cdot NSC/n) \\ &= \sum_{k=1}^{s-1} SC_{i_k} - v(S) + SC_{i_s} + (s+1) \cdot NSC/n \geqslant 0. \end{aligned} \tag{3}$$

We note that $\tilde{x}_i \geqslant 0$ for all $i$ in $N$ and (3) are the conditions stated by Suzuki and Nakayama (1974).

For every vector $y \neq \tilde{x}$ in $X$, there will exist an $i$ in $N$ such that $e(N-i, \tilde{x}) < e(N-i, y)$, which proves that $\tilde{x}$ is the nucleolus. By (2) $e(S, \tilde{x})$ is nonpositive and the core is nonempty. Q.E.D.

In order to interpret this result, we consider, first, games in which the only permissible coalitions are those of size $1, n-1$ and $n$, i.e. pseudo-bargaining games. Davis and Maschler (1965) study these games and prove that the kernel consists of a unique point (their theorem 7.1), namely the nucleolus. These authors further show that under certain conditions the nucleolus is an equity-type vector, i.e. a vector of the form $w + (v(N) - w(N))/n$ where the vector of quotas $w$ solves the system $w(N-i) = v(N-i)$, $i = 1, \ldots, n$.[4] It is immediate that $w_i = (\sum_{j \in N} v(N-j) - (n-1) \cdot v(N-i))/(n-1)$ and that the equity-type vector is equal to $\tilde{x}$. In other words, for games in (PCGI), the players behave as if they were playing a pseudo-bargaining game and were using their quotas $w_i$ as the basis of the negociation. The reason for this behavior can be the following.

For pseudo convex games, $SC_i$ may be understood as a "utopia" (the term is borrowed form Driessen and Tijs 1983b): players in $N-i$ will be better off to exclude $i$ if he/she gets more than $SC_i$; moreover, $SC_i$ is the best a given player $i$ can expect to offer to any coalition to enter. Following this interpretation, the difference $SC_i - y_i$ can be defined as a "utopia loss" that player $i$ suffers w.r.t. $y$. Suppose now that players in $N-i$, once $N-i$ is formed, ask $i$ to pay a fee in order to join. If we rewrite (PI) as,

$$SC_i/(n-1) \geqslant \sum_{j \in N} (SC_j - x_j)/n$$

this condition states that what a player in $N-i$ can expect to receive from $i$ is at least as great as the average utopia loss w.r.t. any imputation. We note that a sufficient con-

[4] This result has been later rediscovered by Owen (1968).

dition for this inequality to hold is,

$$SC_i/(n-1) \geqslant SC_j - x_j, \quad \text{for all } j \text{ in } N,$$

i.e. the utopia loss of every player $j$ in $N - i$ is less than what $i$ can be asked to pay to $j$.

Finally it is interesting to note that for games in (PCGI), $\tilde{x}$ is just the vector which minimizes the distance between the utopia vector and the imputation set, i.e. $\tilde{x}$ solves $\min_{y \in X} \sum_{i \in N} (SC_i - y_i)^2$.[5]

## 3 Some Properties of the Nucleolus

Milnor (1952) was the first to introduce the concept of reasonable outcome. A payoff vector is reasonable if $x_i \leqslant b(i)$ for all $i$ in $N$, where $b(i) = \max_{S \subset N} (v(S) - v(S-i))$. Wesley (1971) proves that each point in the kernel, therefore the nucleolus, is reasonable.

Kikuta (1976) proposes another definition of reasonableness. Whereas Milnor bases his argument upon the maximal marginal contribution of a player to a coalition, Kikuta defines upper-bounds w.r.t. the average of all the marginal contributions of a given player to coalitions of a given size. Formally, an imputation $x$ is reasonable by this definition if $x_i \leqslant b'(i)$, for all $i$ in $N$, where

$$b'(i) = \max_{1 \leqslant k \leqslant n} f_k^i, \quad i = 1, \ldots, k$$

$$f_k^i = \binom{n-1}{k-1}^{-1} \sum_{S \mid \substack{i \in S \\ |S| = k}} (v(S) - v(S-i)), \quad i = 1, \ldots, n, \; k = 1, \ldots, n$$

Kikuta proves that the nucleolus is reasonable for all 4-person simple games with a special property, and for all symmetrical games with nonempty cores. Maschler (1963) proves that the nucleolus is reasonable for all games with coalitions of size $1, n-1$ and $n$. For pseudoconvex games $b(i) = b'(i)$, for all $i$ in $N$. Indeed, if $v \in$ (PCG), $SC_i \geqslant v(S) - v(S-i)$ for all $i \in S \subset N$ and so,

$$\sum_{S \mid \substack{i \in S \\ |S| = k}} SC_i = \binom{n-1}{k-1} SC_i \geqslant \sum_{S \mid \substack{i \in S \\ |S| = k}} (v(S) - v(S-i)).$$

Consequently, the nucleolus for pseudo-convex games, hence for games in (PCGI), is also reasonable by Kikuta's definition.

Another desirable condition for the nucleolus to satisfy is monotonicity. Let $v$ and $w$ be two $n$-person games such that $v(S) = w(S)$ if $S \neq N$ and $w(N) = v(N) + \Delta$ ($\Delta > 0$), and let $\Phi$ be a solution concept which assigns to each $n$-person game $v$ a unique vector

[5] Necessary and sufficient conditions for this result are given in Legros.

$\Phi^v$ in $\mathbb{R}^n$. Then $\Phi$ is monotonic, by Megiddo's (1974) definition if $\Phi_i^w \geqslant \Phi_i^v$ for all $i$ in $N$. The nucleolus is not in general monotonic (Megiddo 1974). For games in (PCGI), however, the nucleolus is monotonic: if $nu^v$ and $nu^w$ are the nucleolus of the games $v$ and $w$ defined above, then $nu_i^w = nu_i^v + \Delta/n$ for all $i$ in $N$.

Recently, Young (1982) has introduced the concept of strong monotonicity of a solution. Let $v^i(S)$ be the contribution of player $i$ to coalition $S$,

$$v^i(S) = \begin{cases} v(S) - v(S-i) & \text{if } i \in S \\ v(S \cup i) - v(S) & \text{if } i \notin S. \end{cases}$$

Consider two games $v$ and $w$ such that $v^i(S) \leqslant w^i(S)$ for all $S$ in $N$. Then a solution $\Phi$ is strongly monotonic if $\Phi_i^w \geqslant \Phi_i^v$. The following example emphasizes that even in (PCGI), the nucleolus may not be strongly monotonic.

Let $|N| = 5$, and two games $v$ and $w$ such that,

$$v(i) = 0,\ i = 1, \ldots, 5; \quad v(S) = 25 \text{ if } |S| = 2; \quad v(S) = 50 \text{ if } |S| = 3;$$

$$v(N-5) = 75; \quad v(N-i) = 74 \text{ if } i \neq 5; \quad v(N) = 100.$$

$$w(i) = 0,\ i = 1, \ldots, 5; \quad w(15) = 25 + \epsilon; \quad w(25) = w(35) = w(45) = 25;$$

$$w(12) = w(13) = w(14) = w(23) = w(24) = w(34) = 24.5; \quad w(S) = 49.5 \text{ if } |S| = 3;$$

$$w(N-5) = 75; \quad w(N-i) = 73.5 \text{ if } i \neq 5; \quad w(N) = 100 + \epsilon.$$

$v$ and $w$ are elements of (PCGI) if $\epsilon \in [0, 1/11]$, and the nucleolus of these games are $nu^w = (20.3 + 0.2\epsilon,\ 20.3 + 0.2\epsilon,\ 20.3 + 0.2\epsilon,\ 20.3 + 0.2\epsilon,\ 18.8 + 0.2\epsilon)$ and $nu^v = (20.2, 20.2, 20.2, 20.2, 19.2)$. So $nu_5^w < nu_5^v$ for $\epsilon \in [0,1/11]$ while $w^5(S) > v^5(S)$ for all $S \subset N$, and the nucleolus is not strongly monotonic.

In this example, the contributions of player 5 to all coalitions are constant – up to $\epsilon$ – in both games $v$ and $w$, while the relative values of the coalitions are quite different in both games. This remark helps to understand why the nucleolus is not strongly monotonic and why imposing strong monotonicity may not be natural. A quite simple example may clarify this point.

Let $|N| = 3$ and two games $v$ and $w$ such that[6]:

$$v(i) = 0,\ i = 1, 2, 3; \quad v(S) = 1 \text{ if } |S| = 2; \quad v(N) = 1$$

$$w(i) = 0,\ i = 1, 2, 3; \quad w(12) = w(13) = 2; \quad w(23) = 200; \quad w(N) = 200.$$

Here $v^i(S) \leqslant w^i(S)$ for all $i$ in $N$ and all coalitions $S$. We have, $nu^v = (1/3, 1/3, 1/3)$ and $nu^w = (0, 100, 100)$. Clearly player 1 would prefer to play $v$ rather than $w$, even if his/her contributions are larger in $w$. Imposing strong monotonicity, i.e. a larger payoff to 1 in $w$ than in $v$, would then hide the fact that player 1 is much weaker in $w$ than in $v$.

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[6] Neither $v$ nor $w$ is in (PCGI) but this is not a problem since we are only concerned with strong monotonicity.

## 4 Some Comparisons

*Definition 2* (Shapley 1971): A game is convex if its characteristic function satisfies $v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$ for all $S, T \subset N$.

*Definition 3* (Sharkey 1982b): Let $B_0 = \phi$ and $B = \{B_1, \ldots, B_k\}$ be an arbitrary partition of $N$. Let $Q = \{Q_1, \ldots, Q_k\}$ be a collection of coalitions such that $Q_i \subset \bigcup_{j=0}^{i-1} B_j$ and $B_i \cup Q_i \neq N$, $i = 1, \ldots, k$. The game $\langle N, v \rangle$ is subconvex if it is true that for all such collections $B$ and $Q$,

$$\sum_{i=k}^{k} (v(B_i \cup Q_i) - v(Q_i)) \leqslant v(N).$$

Sharkey (1982b) interprets subconvexity as an increasing return to coalition formation on average and proves that all subconvex games have large cores. Let us denote by (CG) and (SCG) the classes of convex and subconvex games.

*Theorem 2:* (CG) $\subset$ (SCG) $\subset$ (PCG).

*Proof:* Let $v \in$ (CG) and consider a pair $(B, Q)$ of collections of coalitions which satisfy the conditions of Definition 3. We show by induction that,

$$\sum_{i=1}^{t} (v(B_i \cup Q_i) - v(Q_i)) \leqslant v\left(\bigcup_{i=1}^{t} B_i\right). \tag{4}$$

(4) is obviously true for $t = 1$. At $t + 1$, we have,

$$\sum_{i=1}^{t+1} v(B_i \cup Q_i) = \sum_{i=1}^{t} v(B_i \cup Q_i) + v(B_{t+1} \cup Q_{t+1}).$$

By convexity of $v$,

$$v\left(\bigcup_{i=1}^{t} B_i\right) + v(B_{t+1} \cup Q_{t+1}) \leqslant v\left(\bigcup_{i=1}^{t} B_i \cup (B_{t+1} \cup Q_{t+1})\right) + v\left(\bigcup_{i=1}^{t} B_i \cap (B_{t+1} \cup Q_{t+1})\right). \tag{5}$$

Since $Q_{t+1} \subset \bigcup_{i=1}^{t} B_i$ by Definition 3, (4) and (5) imply,

$$\sum_{i=1}^{t+1} v(B_i \cup Q_i) \leqslant v\left(\bigcup_{i=1}^{t+1} B_i\right) + \sum_{i=1}^{t+1} v(Q_i).$$

So, (4) is true for all $t = 1, \ldots, k$, and because $B$ is a partition of $N$, $v\left(\bigcup_{i=1}^{k} B_i\right) = v(N)$, which proves that $v \in$ (SCG).

Consider now the $(2^n - 1)$ partitions $B_S = \{S, N-S\}$ where $S \subset N$. Let $Q_S = \{\phi, S-i\}$ where $i \in S$. $B_S$ and $Q_S$ clearly satisfy Definition 3. So, if $v \in$ (SCG),

$$v(S) + v(N-i) - v(S-i) \leqslant v(N). \tag{6}$$

(6) is true for all $S \subset N$, and so $v \in$ (PCG). Q.E.D.

*Corollary:* (CG) = (SCG) = (PCG) if $|N| = 3$.

*Proof:* Note that (CG) = (PCG) if $|N| = 3$ and use Theorem 2. Q.E.D.

We show now that (CG) and (SCG) are distinct from (PCGI) with a sequence of examples.

*Claim 1:* (CG) − (PCGI) $\neq \phi$.

Consider the 4-person game, $v(i) = 0, i = 1,2,3,4$; $v(S) = 5$ if $|S| = 2$; $v(123) = 16$; $v(124) = 12$; $v(134) = 10$; $v(234) = 14$; $v(N) = 26$.

This game is convex but $SC_4 = 10 < 39/2 = -3/4 \cdot NSC$ and so $v \notin$ (PCGI).

*Claim 2:* (CG) $\cap$ (PCGI) $\neq \phi$.

Let $|N| = 4$ and $v(i) = 0, i = 1,2,3,4$; $v(S) = 2/5$ if $|S| = 2$; $v(S) = 1$ if $|S| = 3$ and $v(N) = 8/5$.

This game is convex and (PI) is satisfied: for all $i$, $SC_i = 3/5 = -3/4 \cdot NSC$.

*Claim 3:* (PCGI) − (SCG) $\neq \phi$.

Let $|N| = 5$ and $v(i) = 0$ for all $i$ in $N$; $v(S) = 25$ if $|S| = 2$; $v(123) = v(124) = 25$; $v(S) = 50$ if $|S| = 3$ and $S \notin \{123, 124\}$; $v(N-5) = 75$; $v(N-i) = 74$ if $i \in N-5$; $v(N) = 100$.

$SC_5 = 25$ and $SC_i = 26$ if $i \neq 5$. So, $-4/5 \cdot NSC = 23.2 < SC_i$ for all $i$ in $N$, and $v$ satisfies (PI). It is routine to check that $v$ is pseudo-convex, so $v \in$ (PCGI). Consider now the families $B = \{1, 23, 4, 5\}$ and $Q = \{\phi, 1, 12, 1\}$; $B$ and $Q$ satisfy the conditions of Definition 3, but we have,

$$\sum_{i=1}^{4} (v(B_i \cup Q_i) - v(Q_i)) = v(123) + v(124) + v(15) - v(12) = 102 > v(N)$$

and $v \notin$ (SCG) which proves the claim.

We consider now the class $\widetilde{Q}$ proposed by Driessen and Tijs (1983a). This class is defined as follows,

$$\widetilde{Q} = \{v \in Q \text{ s.t. } SC(N) - v(N) \leqslant SC(S) - v(S), \text{ for all } S \subset N\}$$

where

$$Q = \{v \text{ s.t. } a \leqslant SC,\ a(N) \leqslant v(N) \leqslant SC(N)\}$$

$$a_i = \max_{S \mid i \in S} (v(S) - SC(S - i)), \quad i = 1, \ldots, n.$$

Theorems 4.1 and 4.7 of Driessen and Tijs (1983a) establish that the nucleolus of all games in $\tilde{Q}$ is equal to the vector $SC + NSC/n$. The following theorem and remark have been obtained by Driessen and Tijs and the author (private communication).

*Theorem 3:* $\tilde{Q} \cap (\text{PCG}) = \tilde{Q} \cap (\text{PCGI}) = EG = \{v \text{ s.t. } v(S) = \frac{s-1}{n-1} \cdot v(N), S \subset N\}$.

*Proof:* If $v \in (\text{PCG})$, $SC_i \geqslant v(S) - v(S-i)$ for all $i \in S$. This inequality may be rewritten as $SC(S) - v(S) \geqslant SC(S-i) - v(S-i)$. Since this is true for every coalition, we deduce that if $v \in (\text{PCG})$,

$$SC(S) - v(S) \geqslant SC(T) - v(T) \quad \text{for all } T \subset S \subset N. \tag{7}$$

Now, if $v \in \tilde{Q}$

$$SC(N) - v(N) \leqslant SC(T) - v(T) \quad \text{for all } T \subset N. \tag{8}$$

Consequently, (7) and (8) imply, if $v \in (\text{PCG}) \cap \tilde{Q}$

$$SC(T) - v(T) = SC(N) - v(N) \quad \text{for all } T \subset N. \tag{9}$$

(9) defines the class of games $EG$ which was first introduced by Driessen and Tijs. Hereafter we characterize these games. Let $T = \{j\}$; then by (9) $SC_j = SC(N) - v(N)$, for all $j$ in $N$ and so $SC_j = SC_k$ for all $j \neq k$, i.e. $v(N-j) = v(N-k) = z$ for all $j \neq k$. It follows that $v(N) - z = (n-1)v(N) - nz$ and $z = \frac{n-2}{n-1} \cdot v(N)$. If we insert this value in (9), we get $v(S) = \frac{s-1}{n-1} \cdot v(N)$. So, $\tilde{Q} \cap (\text{PCG}) \subset EG$. Finally for $v \in EG$, it is immediate that $v \in \tilde{Q} \cap (\text{PCG})$, and so $\tilde{Q} \cap (\text{PCG}) = EG$. If $v \in EG$, $SC_i = v(N)/(n-1)$ $> v(N)/n = \frac{1-n}{n} \cdot NSC$ and $v$ satisfies (PI). Q.E.D.

*Remark:* It is easy to show that as $n$ becomes large enough, (PCGI) is equal to $EG$. Indeed, if $v \in (\text{PCGI})$, by (7) and (PI),

$$\frac{n-1}{n}(SC(N) - v(N)) \leqslant SC_i \leqslant SC(N) - v(N).$$

So, if $n$ becomes large, $SC_i$ is equal to $SC(N) - v(N)$, and from (7) we deduce that $SC(S) - v(S)$ is also equal to $SC(N) - v(N)$, which proves that for large $n$, each game in (PCGI) is in $EG$.

## 5 Concluding Remarks

We show in this paper that the SCRB method can be supported by a normative concept whenever the cost function defines a saving game which is in (PCGI). Moreover, we propose in Section 2 an interpretation of the result in terms of utopia and entrance fees. It is interesting to make the analogy between this interpretation and Moulin's ALDB (Auctioning the leadership with differentiated bids) mechanism.

In the ALDB procedure, the players act noncooperatively and first choose as the leader the player who has the lowest bid (i.e. the amount of money a player wants to get from each other player to become the leader). Once the leader proposes a decision (i.e. chooses a given state among the set of possible alternatives together with a vector of transfers), the players vote unanimously to accept the decision. If the decision is rejected, the leader is eliminated from the decision process and another leader is chosen. This procedure is repeated until a decision is unanimously approved. Moulin (1981) shows that if the players behave in a sophisticated way, this mechanism implements the SCRB vector and the optimal bid for each player is exactly –NSC/n, where the function $v$ is defined by $v(S) = \max_{a \in A} \sum_{i \in S} u_i(a)$, all $S \subset N$, with $u_i$ being the utility function and $A$ being the set of possible states.

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