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Computation of the Nucleolus of Some Bilateral Market Games[1]

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Abstract: This paper gives a simple algorithm for computing the nucleolus of bilateral markets with two complementary commodities.

Introduction

Maschler, Peleg and Shapley give a geometric characterization of the central place of the nucleolus in the imputation space. Their method is related to the algorithm of Kopelowitcz which consists of solving a sequence of linear programs. Papers of Brune, Owen, Kohlberg, Bruynell and Grotte give some general results concerning the properties and the computation of the nucleolus. However, little is known about the evolution of the nucleolus within a given class of games; exceptions are results of Littlechild for special cost games, of Galil for weighted majority games, and of Driessen and Tijs for special economic games. The purpose of this paper is to compute the nucleolus of the following class of games.

We consider the class of cooperative games with side-payments and whose characteristic function is,

$$v(S) = |S \cap P| \wedge \lambda \cdot |S \cap Q| \quad \text{all } S \subset P \cup Q \tag{1.1}$$

where $P \cup Q$ is the set of players
$\wedge$ is the min operator: $a \wedge b = \min(a, b)$
$\lambda \in \mathbb{R}_+^*$.

This is typically a generalization of the well known gloves game (take $\lambda = 1$). We can consider for instance that P is a set of manufacturers, each of whom owns a machine, and that Q is a set of workers each willing to work a certain amount of time. Then $v(S)$ represents the net profit that the coalition S can get with the output produced by

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the cooperation of the $|S \cap P|$ manufacturers and the $|S \cap Q|$ workers, λ being a technical coefficient which illustrates the complementarity between machines and worker-hours.[3] The problem is then to allocate the total profit $v(P \cup Q)$ between manufacturers and workers.

The set of imputations is the set of Pareto optimal and individually rational allocations;

$$I_v = \{x \in \mathbb{R}^{|P \cup Q|} | x_i \geq 0 \quad \text{and} \quad \sum_{i \in P \cup Q} x_i = v(P \cup Q)\} \tag{1.2}$$

We introduce the following notation and definitions. Capital letters will refer to coalitions and small letters to numbers. When no confusion is possible we denote $s = |S|$ the number of elements of the coalition S. We do not distinguish between a coalition and its profile, i.e. we write $T \equiv (t, \sigma(t))$ where $t \equiv |T \cap P|$ and $\sigma(t) \equiv |T \cap Q|$. For every coalition T, the excess of T with respect to (wrt) the imputation x is the difference $v(T) - \sum_{i \in T} x_i$ and is denoted either $e(T, x)$ or $e_x(t, \sigma(t))$.

Let $\theta(x)$ be the vector of $\mathbb{R}^{2^{p+q}-1}$ for which the components are the excesses ranked in decreasing order. An imputation x is preferred to an imputation y if and only if (iff) $\theta(x)$ is before $\theta(y)$ in the lexicographic ordering on $\mathbb{R}^{2^{p+q}-1}$.[4] The nucleolus is the vector which is preferred to every other imputation. Schmeidler proves that the nucleolus always exists, is unique, is an element of every ϵ-core and is a continuous function of the characteristic function. We denote the nucleolus by *nu*.

A vector x in $\mathbb{R}^{p+q}$ s.t. $x_i = \alpha$ for all i in P and $x_j = \beta$ for all j in Q is denoted $((\alpha)^p, (\beta)^q) = x$. We denote by $X_{p,q}$ the set of imputations which take this form,

$$X_{p,q} = \{x \in I_v | x = ((\alpha)^p, (\beta)^q)\}. \tag{1.3}$$

2 Preliminary Results

Lemma 2.1: $nu \in X_{p,q}$.

Proof: Maschler and Peleg prove that symmetric players receive equal payoffs in the kernel. Because the nucleolus is an element of the kernel, the result follows. Q.E.D.

Lemma 2.2: $e(T, nu) \leq 0$ for all coalitions T.

Proof: It suffices to show that the core is nonempty. If $\lambda \leq p/q$, consider the vector $x = ((0)^p, (\lambda)^q)$. Then $e_x(t, \sigma(t)) = t \wedge \lambda\sigma(t) - \lambda\sigma(t)$ for all coalitions $(t, \sigma(t))$, and so $e_x(t, \sigma(t)) \leq 0$. Similarly, if $\lambda \geq p/q$, consider the vector $x = ((1)^p, (0)^q)$. Then $e_x(t, \sigma(t)) = t \wedge \lambda\sigma(t) - t \leq 0$. It is in fact possible to show that every subgame also has a nonempty core, i.e. that (1.1) defines a market game. Q.E.D.

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[3] This interpretation is borrowed from Maschler.

[4] I.e. there exists i_0 such that (s.t.) $\theta_i(x) = \theta_i(y)$ for all $i < i_0$ and $\theta_{i_0}(x) < \theta_{i_0}(y)$.

Let (A_+, A_-) be the following partition of $2^{P \cup Q}$,

$$A_+ = \{(t, \sigma(t)) \in [0,p] \times [0,q] \mid t \geqslant \lambda\sigma(t)\} \qquad (2.1)$$
$$A_- = \{(t, \sigma(t)) \in [0,p] \times [0,q] \mid t < \lambda\sigma(t)\}.$$

Lemma 2.3: The excesses wrt $x = ((\alpha)^p, (\beta)^q)$ are the following

(i) If $\lambda \leqslant p/q$, $e_x(t, \sigma(t)) = \{p\sigma(t)/q - t\}\alpha$ if $(t, \sigma(t)) \in A_+$
$= \{p\sigma(t)/q - t\}\alpha + t - \lambda\sigma(t)$ if $(t, \sigma(t)) \in A_-$.

(ii) If $\lambda \geqslant p/q$, $e_x(t, \sigma(t)) = \{p\sigma(t)/q - t\}\alpha + \sigma(t)(\lambda - p/q)$ if $(t, \sigma(t)) \in A_+$
$= \{p\sigma(t)/q - t\}(\alpha - 1)$ if $(t, \sigma(t)) \in A_-$.

Proof: For all coalitions $(t, \sigma(t))$ and all vectors $x = ((\alpha)^p, (\beta)^q)$, $e_x(t, \sigma(t)) = t \wedge \lambda\sigma(t) - t\alpha - \sigma(t)\beta$. Because $q\beta = p \wedge \lambda q - p\alpha$, some simple transformations yield the result. Q.E.D.

Lemma 2.4

(i) Let $\lambda \leqslant p/q$. If there exists $(t, \sigma(t)) \in A_+$ s.t. $\frac{p}{q}\sigma(t) - t > 0$, then $nu = ((0)^p, (\lambda)^q)$.

(ii) Let $\lambda \geqslant p/q$. If there exists $(t, \sigma(t)) \in A_-$ s.t. $\frac{p}{q}\sigma(t) - t < 0$, then $nu = ((1)^p, (0)^q)$.

(iii) Otherwise, the nucleolus is defined by the equality

$$\max_{\substack{S \in A_+ \\ |S \cap P| \neq p}} e(S, x) = \max_{S \in A_-} e(S, x).$$

Proof: (i) and (ii). A direct consequence of Lemma 2.2.

(iii) By Lemma 2.3, it is clear that when α rises, the excesses in A_+ and A_- vary in opposite directions. It follows that α must be such that the equality in Lemma 2.4 is true. First, it is immediate that $(p, q) \in A_+$ and that $e_x(p, q) = 0$ for all $x \in X_{p,q}$. We can then restrict our attention to coalitions in A_+ such that $|S \cap P| \leqslant p - 1$. Now, suppose that $e(S_0, x) > e(T_0, x)$, where $S_0 \in A_+$ and $T_0 \in A_-$ and where $e(S_0, x) \geqslant e(S, x)$ for all $S \in A_+$ and $e(T_0, x) \geqslant e(T, x)$ for all $T \in A_-$. By Lemma 2.1, the excesses wrt the nucleolus are given by Lemma 2.3. Let $y = \{(\alpha')^p, (\beta')^q\} \in X_{p,q}$, where, for an appropriate choice of $\alpha' > \alpha$, we have, $e(S_0, x) > e(S_0, y) = e(T_0, y) > e(T_0, x)$. Precisely, take $\alpha' = \alpha + [e(S_0, x) - e(T_0, x)]/[p\{\sigma(t_0) - \sigma(s_0)\}/q - t_0 + s_0]$.[5] This proves that x cannot be the nucleolus of the game. Q.E.D.

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[5] We check that the RHS of this equality is indeed greater than α (apply the definitions of A_- and A_+ and Lemma 2.2).

We consider the case $q \geqslant p$. This assumption does not imply a loss of generality since if $q < p$ we have for all coalitions $(t, \sigma(t))$, $v(t, \sigma(t)) = t \wedge \lambda\sigma(t) = \lambda\{\sigma(t) \wedge t/\lambda\}$ i.e. $v(t, \sigma(t)) = \lambda w(\sigma(t), t)$. Clearly v and w are equivalent and so $nu^v = \lambda nu^w$, where nu^v and nu^w are the nucleolus of the games v and w respectively.

Furthermore, we shall present the complete proofs of the results only for the case $\lambda \leqslant p/q$, since the proofs in the case $\lambda \geqslant p/q$ are similar to the proofs for $\lambda \leqslant p/q$.

For all $t \in [0, p]$ we define $\sigma^*(t)$ to be the integer in $[0, q]$ such that $(t, \sigma^*(t)) \in A_+$ and $(t, \sigma^*(t) + 1) \in A_-$. It is clear that $\sigma^*(t)$ is unique for all t.

Lemma 2.5: Let $x \in X_{p,q}$, then for every $t \in [0, p]$,

$$e_x(t, \sigma^*(t)) \geqslant e_x(t, \sigma(t)) \qquad \text{for all } \sigma(t) \leqslant \sigma^*(t)$$

$$e_x(t, \sigma^*(t) + 1) \geqslant e_x(t, \sigma(t)) \quad \text{for all } \sigma(t) \geqslant \sigma^*(t) + 1.$$

Proof: A direct consequence of Lemma 2.3. Q.E.D.

When q is proportional to p, i.e. when $q = pk$, $k \in \mathbb{N}$, we obtain a quite simple characterization of the nucleolus. We first consider this "k-market" before returning to the general case.

3 The k-market

Theorem 3.1: Let $q = pk$, $k \in \mathbb{N}$,

(i) If $\lambda \in [0, (p-1)/(q-k+1)]$, $nu = ((0)^p, (\lambda)^q)$.

(ii) If $\lambda \in [(p-1)/(q-k+1), p/q]$,

$$nu = \left(\left(-\frac{k}{2}\{p - 1 - \lambda(q-k+1)\} \right)^p, \left(\frac{1}{2}\{p - 1 - \lambda(q-k-1)\} \right)^q \right).$$

(iii) If $\lambda \in [p/q, (p-1)/(q-k-1)]$,

$$nu = \left(\left(1 - \frac{k}{2}\{p - 1 - \lambda(q-k-1)\} \right)^p, \left(\frac{1}{2}\{p - 1 - \lambda(q-k-1)\} \right)^q \right).$$

(iv) If $\lambda \geqslant (p-1)/(q-k-1)$, $nu = ((1)^p, (0)^q)$.

Proof: (i) Since $q > p(k-1)$, we have $(p-1)/(q-k+1) > 1/(k-1)$. Therefore, $(1, k-1)$ and $(p-1, q-k+1)$ belong to A_+. By Lemma 2.3, we have for all $x = ((\alpha)^p, (\beta)^q) \in X_{p,q}$, $e_x(1, k-1) = \{p(k-1)/q - 1\}\alpha = -p/q$ and $e_x(p-1, q-k+1) = \{p(q-k+1)/q - p + 1\}\alpha = p\alpha/q$. Lemma 2.2 insures that $\alpha = 0$ if x is the nucleolus of the game.

(ii) Since $\lambda \leqslant p/q = 1/k$, $(t, kt) \in A_+$ for all t. Suppose now that $(t, kt+1) \in A_+$; Then $t \geqslant \lambda(kt+1)$ which implies by the hypothesis on λ that $t \geqslant (p-1)(kt+1)/(q-k+1)$, i.e. $t \geqslant p-1$ since $q = pk$. Because the coalition $(p, pk+1)$ does not exist, the only possibility is $t = p-1$; but if $(p-1, k(p-1)+1) \in A_+, p-1 \geqslant \lambda\{k(p-1)+1\}$ and so $\lambda = (p-1)/(q-k+1)$ since $\lambda \geqslant (p-1)/(q-k+1)$ by hypothesis. In this case, $\alpha = 0$ by (i). Otherwise, we deduce that $(t, kt+1) \in A_-$, which means that $\sigma^*(t) = kt$ for all t. Because $e_x(t, kt) = 0$ for all t and all $x \in X_{p,q}$, if we note $\sigma^{**}(t) = kt-1$, we have by Lemma 2.3,

$$e_x(t, \sigma^{**}(t)) = -p\alpha/q \tag{3.1}$$

$$e_x(t, \sigma^*(t)+1) = p\alpha/q + t - \lambda(kt+1). \tag{3.2}$$

Since $1 - \lambda k \geqslant 0$, we deduce that,

$$p - 1 = \underset{t \in [0, p-1]}{\operatorname{argmax}}\ e_x(t, \sigma^*(t)+1).$$

By Lemma 2.4, we finally obtain the desired expression of the nucleolus by replacing t by $p-1$ in (3.2) and equating (3.2) with (3.1).

(iii) and (iv) are proved in the same way. Q.E.D.

Remark 3.2: We note that the k-market is the p-replication of a market with one agent of type P and k agents of type Q. It is easy to compute the nucleolus of such games (see Legros) and the reader can verify that we have $nu = (\lambda k/2, (\lambda/2)^k)$ if $\lambda \in [0, 1/k]$, $nu = (1 - k\{\lambda(k-1)+1\}/2, \quad (\{1-\lambda(k-1)\}/2)^k)$ if $\lambda \in [1/k, 1/(k-1)]$, and $nu = (1, (0)^k)$ if $\lambda \geqslant 1/(k-1)$. We observe that the player of type P always gets half of the sum of the incremental contributions of the k players of type Q, i.e. the amount $k\{v(1,k) - v(1,k-1)\}/2$. This result can be explained by the monopoly power of the player of type P. From Theorem 3.1 we deduce that this monopoly power is weakened in the p-replicated game: every player in P gets less than half of the sum of the incremental contributions of k players of type Q.

4 The General Case

We denote by $[\![x]\!]$ the integer part of a real x and $k = [\![q/p]\!]$. We assume henceforth that $q \geqslant pk+1$.

Lemma 4.1: Let $q \geqslant pk+1$,

(i) If $\lambda \in [0, (p-1)/(q-k)]$, $nu = ((0)^p, (\lambda)^q)$.

(ii) If $\lambda \geqslant (p-1)/(q-k-1)$, $nu = ((1)^p, (0)^q)$.

Proof: (i) If $q > pk$, $(p-1)/(q-k) < 1/k$ and the coalitions $(1,k)$ and $(p-1, q-k)$ belong to A_-. From Lemma 2.3, for all $x \in X_{p,q}$, $e_x(1,k) = -e_x(p-1,q-k) = (pk/q - 1)\alpha$. Since $pk/q - 1 \neq 0$, by Lemma 2.2, we get $\alpha = 0$.

(ii) The same argument with $(1, k+1)$ and $(p-1, q-k-1)$. Q.E.D.

We shall now consider the case $\lambda \in [(p-1)/(q-k), (p-1)/(q-k-1)]$.

Lemma 4.2: Let $\delta(t) = \sigma^*(t+1) - \sigma^*(t)$. Then $\delta(t) \in \{k, k+1\}$ for all $t \in [0, p-1]$.

Proof: Let $r \geqslant 0$ and suppose that $\delta(t) = k - r$. We have,

$$t \geqslant \lambda\sigma^*(t) = \lambda\sigma^*(t+1) - \lambda(k-r)$$

and so, $t + 1 \geqslant \lambda\sigma^*(t+1) - \lambda(k-r) + 1$. But $t + 1 < \lambda\{\sigma^*(t+1) + 1\}$ by definition of $\sigma^*(t+1)$. By subtracting these two inequalities we get, $\lambda > 1/(k-r+1)$ which is compatible with $\lambda < 1/k$ only if $r = 0$. So, $\delta(t) = k$ in this case.

Suppose now that $\delta(t) = k + r$, $r \geqslant 0$. Because $(t, \sigma^*(t) + 1) \in A_-$ and $(t+1, \sigma^*(t+1)) \in A_+$, we have, $t + 1 < \lambda\{\sigma^*(t+1) - k - r + 1\} + 1$ and $t + 1 \geqslant \lambda\sigma^*(t+1)$. By subtracting these two inequalities we obtain $\lambda < 1/(k+r-1)$ which contradicts $\lambda > 1/(k+1)$ for $r \geqslant 2$. Then $r = 0$ or $r = 1$ and this proves the Lemma. Q.E.D.

Lemma 4.3

$$e_{nu}(t, \sigma^*(t)) - e_{nu}(t+1, \sigma^*(t+1)) \geqslant 0 \quad \text{if } \delta(t) = k$$
$$\leqslant 0 \quad \text{if } \delta(t) = k+1$$

$$e_{nu}(t, \sigma^*(t) + 1) - e_{nu}(t+1, \sigma^*(t+1) + 1) \geqslant 0 \quad \text{if } \delta(t) = k+1$$
$$\leqslant 0 \quad \text{if } \delta(t) = k.$$

Proof: Let $\lambda \in [(p-1)/(q-k), p/q]$, then, $e_{nu}(t, \sigma^*(t)) - e_{nu}(t+1, \sigma^*(t+1)) = -\{p\delta(t)/q - 1\}\alpha$. Because $1/(k+1) < p/q < 1/k$ and $\alpha \geqslant 0$ the result follows.

Now, $e_{nu}(t, \sigma^*(t) + 1) - e_{nu}(t+1, \sigma^*(t+1) + 1) = -\{p\delta(t)/q - 1\}\alpha - 1 + \lambda\delta(t)$. if $\delta(t) = k$, $pk/q - 1 < 0$ and $\lambda k - 1 < 0$, the difference must be nonpositive since by Pareto-optimality, $\alpha \leqslant 1 \leqslant (1 - \lambda k)/(1 - pk/q)$. If $\delta(t) = k + 1$, the difference is exactly equal to $-e_{nu}(1, k+1)$ which is nonegative since $(1, k+1) \in A_-$ and by Lemma 2.2.

When $\lambda \in [p/q, (p-1)/(q-k-1)]$ the same arguments yield the result. Q.E.D.

Lemma 4.3 allows us to deduce the expression of the nucleolus in two special cases. We first need an intermediate result.

Lemma 4.4: Let $l = |\{t \in [1, p-2] \text{ s.t. } \delta(t) = k\}|$, then $l = p(k+1) - q - 1$.

Proof: We have,

$$\sum_{t=1}^{p-2} \delta(t) = \sum_{t|\delta(t)=k} k + \sum_{t|\delta(t)=k+1} (k+1)$$
$$= lk + (p-2-l)(k+1)$$
$$= (p-2)k + p - 2 - l.$$

Since $\lambda < (p-1)/(q-k-1)$ and $\lambda > (p-1)/(q-k)$, $\sigma^*(p-1) = q-k-1$. But $\sigma^*(p-1)$ can be written as, $\sigma^*(p-1) = \sigma^*(1) + \sum_{t=1}^{p-2} \delta(t)$. Since $\sigma^*(1) = k$, we deduce that $(p-2)k + p - 2 - l = q - 2k - 1$, i.e. $l = p(k+1) - q - 1$. Q.E.D.

Theorem 4.5

(i) Let $\lambda \in ((p-1)/(q-k), p/q]$,

if $q = pk + 1$, $nu = ((q\{\lambda(q-k) - p + 1\}/2)^p, (\{\lambda - (\lambda q - p)(p-1)\}/2)^q)$,

if $q = pk + p - 1$, $nu = ((q\{\lambda(k+1) - 1\}/2)^p, (\{p - \lambda(q-1)\}/2)^q)$.

(ii) Let $\lambda \in [p/q, (p-1)/(q-k-1))$,

if $q = pk + 1$, $nu = ((\{1 + k(\lambda q - p)\}/2)^p, (p\{1 - k(\lambda q - p)\}/(2q))^q)$,

if $q = pk + p - 1$,

$nu = ((\{1 + k(\lambda q - p)/(p-1)\}/2)^p, (p\{1 - k(\lambda q - p)/(p-1)\}/(2q))^q)$.

Proof: As before, the proof of (ii) parallels the proof of (i) and is not presented. (i) Let $q = pk + 1$. Lemma 4.4 implies that $l = p - 2$, and so $\delta(t) = k$ for all $t \in [0, p-2]$. Lemma 4.3 implies that $e_{nu}(1, k) = \max_{\substack{(t,\sigma(t)) \in A_+ \\ t \neq p}} e_{nu}(t, \sigma(t))$ and $e_{nu}(p-1, q-k) = \max_{(t,\sigma(t)) \in A_-} e_{nu}(t, \sigma(t))$. By Lemma 4.4, $e_{nu}(1,k) = e_{nu}(p-1, q-k)$ defines the nucleolus and we finally obtain the vector of the Theorem 4.5.

Let $q = pk + p - 1$. Then $l = 0$ in Lemma 4.4 and $\delta(t) = k + 1$ for all $t \in [0, p-2]$. The nucleolus is defined by the equality $e_{nu}(p-1, q-k-1) = e_{nu}(1, k+1)$ and we find the vector given by the theorem. Q.E.D.

Remark 4.6: The two situations considered by Theorem 4.5 are when a player of type Q enters a k-market (if $q = pk + 1$) and when a player of type Q exits a $(k+1)$-market (when $q = pk + p - 1$). A simple comparison of the nucleolus of the corresponding games (before/after exit or entrance) shows that the payoff of each player in P increases (respectively decreases) as a player of type Q enters (exits from) a k-market.

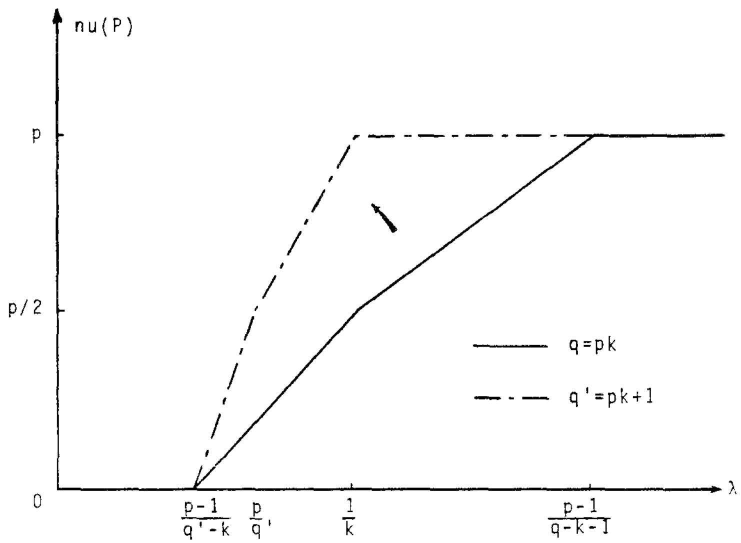

Fig. 1. Entrance in a k-market

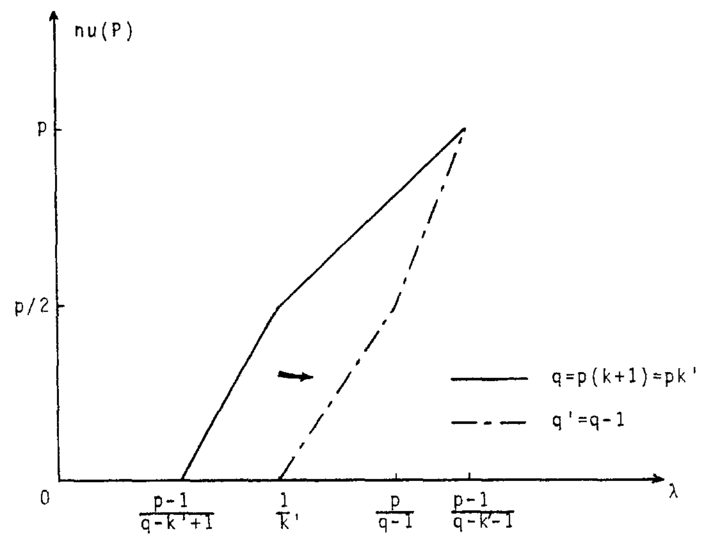

Fig. 2. Exit from a $(k+1)$-market

The evolution of the payments of the players of type P wrt the nucleolus as λ varies is illustrated in the Fig. 1 and 2 above.

We now restrict our attention to the case $\lambda \in ((p-1)/(q-k), p/q]$. In order to simplify the writing, we use the following notation: if $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, we denote $\mu = \lambda^{-1}$, $\bar{\mu} = \underline{\lambda}^{-1}$ and $\underline{\mu} = \bar{\lambda}^{-1}$.

Lemma 4.7

(i) For all t, $\llbracket \bar{\mu}t \rrbracket = \llbracket \underline{\mu}t \rrbracket$ or $\llbracket \bar{\mu}t \rrbracket = \llbracket \underline{\mu}t \rrbracket + 1$.

(ii) If $\llbracket \underline{\mu}t \rrbracket = \llbracket \bar{\mu}t \rrbracket$, $\sigma^*(t) = \llbracket \underline{\mu}t \rrbracket$.

Proof: (i) We know that $q = pk + r$ with $1 \leqslant r \leqslant p - 1$. Then

$$\begin{aligned}\llbracket \bar{\mu}t \rrbracket &= \llbracket (q-k)t/(p-1) \rrbracket = \llbracket kt + rt/(p-1) \rrbracket \\ &= kt + \llbracket rt/(p-1) \rrbracket\end{aligned}$$

$$\begin{aligned}\llbracket \underline{\mu}t \rrbracket &= \llbracket qt/p \rrbracket = \llbracket kt + rt/p \rrbracket \\ &= kt + \llbracket rt/p \rrbracket.\end{aligned}$$

Because $rt/(p-1) - rt/p = rt/\{p(p-1)\}$ and because $rt \leqslant p(p-1)$, we deduce that the difference can be equal at most to 1. It follows that $\llbracket \bar{\mu}t \rrbracket \leqslant \llbracket \underline{\mu}t \rrbracket + 1$.

(ii) From the inequalities $\llbracket \underline{\mu}t \rrbracket \leqslant \underline{\mu}t \leqslant \bar{\mu}t$, we deduce that $(t, \llbracket \underline{\mu}t \rrbracket) \in A_+$. Let t be such that $\llbracket \underline{\mu}t \rrbracket = \llbracket \bar{\mu}t \rrbracket$. Then $\llbracket \underline{\mu}t \rrbracket + 1 = \llbracket \bar{\mu}t \rrbracket + 1 > \bar{\mu}t \geqslant \underline{\mu}t$, and $(t, \llbracket \underline{\mu}t \rrbracket + 1) \in A_+$. So, $\sigma^*(t) = \llbracket \underline{\mu}t \rrbracket = \llbracket \bar{\mu}t \rrbracket$. Q.E.D.

Lemma 4.8: Let a and b be two nonegative reals; then,

(i) $\llbracket a + b \rrbracket \geqslant \llbracket a \rrbracket + \llbracket b \rrbracket$

(ii) $\llbracket a - b \rrbracket \leqslant \llbracket a \rrbracket - \llbracket b \rrbracket$ where $a \geqslant b$.

Proof: (ii) $\llbracket a + b \rrbracket$ is the greatest integer less than $a + b$ and $\llbracket a \rrbracket + \llbracket b \rrbracket$ is an integer less than $a + b$, so $\llbracket a \rrbracket + \llbracket b \rrbracket \leqslant \llbracket a + b \rrbracket$.

(ii) Take $a = a' - b$ in (i) with $a' \geqslant b$. Then $\llbracket a' \rrbracket \geqslant \llbracket a' - b \rrbracket + \llbracket b \rrbracket$ which proves (ii). Q.E.D.

Lemma 4.9: Let $M = \{t \in [0, p] \mid \llbracket \underline{\mu}t \rrbracket = \llbracket \bar{\mu}t \rrbracket - 1\}$. Then $M \neq \emptyset$.

Proof: We show that $p - 1 \in M$. Indeed, $\llbracket \underline{\mu}(p-1) \rrbracket = \llbracket q(p-1)/p \rrbracket$, and by Lemma 4.8, $\llbracket \underline{\mu}(p-1) \rrbracket < q - \llbracket q/p \rrbracket = q - k$. The inequality is strict since $q - q/p \leqslant q - k - 1/p < q - k$. Now, $q - k = \llbracket \bar{\mu}(p-1) \rrbracket$, and so $\llbracket \underline{\mu}(p-1 \rrbracket < \llbracket \bar{\mu}(p-1) \rrbracket$ which implies by Lemma 4.7 that $\llbracket \bar{\mu}(p-1) \rrbracket = \llbracket \underline{\mu}(p-1) \rrbracket + 1$, i.e. $p - 1 \in M$. Q.E.D.

Lemma 4.10: For all $t, s \in [0, p]$ with $t \geqslant s$, $\llbracket \bar{\mu}(t-s) \rrbracket$ is equal to $\llbracket \bar{\mu}t \rrbracket - \llbracket \bar{\mu}s \rrbracket$ or to $\llbracket \bar{\mu}t \rrbracket - \llbracket \bar{\mu}s \rrbracket - 1$.

Proof. From Lemma 4.8, $\llbracket \bar{\mu}(t-s) \rrbracket \leqslant \llbracket \bar{\mu}t \rrbracket - \llbracket \bar{\mu}s \rrbracket$. Now by definition of the integer part, $\llbracket \bar{\mu}(t-s) \rrbracket > \bar{\mu}(t-s) - 1$ and so $\llbracket \bar{\mu}(t-s) \rrbracket > \bar{\mu}t - \bar{\mu}s - 1 \geqslant \llbracket \bar{\mu}t \rrbracket - \llbracket \bar{\mu}s \rrbracket - 2$, which implies that $\llbracket \bar{\mu}(t-s) \rrbracket \geqslant \llbracket \bar{\mu}t \rrbracket - \llbracket \bar{\mu}s \rrbracket - 1$ and proves the Lemma. Q.E.D.

We are now able to prove an important result.

Theorem 4.11: Let $t^* = \min \{t \mid t \in \operatorname{argmax}_{t\in M} t/[\![\bar{\mu}t]\!]\}$. Then the following equivalence is true,

$$nu = ((0)^p, (\lambda)^q) \Leftrightarrow \lambda \leqslant t^*/[\![\bar{\mu}t^*]\!].$$

Proof: (Necessity) Let $\lambda > t^*/[\![\bar{\mu}t^*]\!]$. Then $\sigma^*(t) = [\![\underline{\mu}t]\!]$ for all $t \in M$. Indeed, $(t^*, [\![\underline{\mu}t^*]\!] + 1) \in A_-$ from the hypothesis on λ, and because $t^*/[\![\bar{\mu}t^*]\!] \geqslant t/[\![\bar{\mu}t]\!]$ for all $t \in M$, we have $(t, [\![\bar{\mu}t]\!]) \in A_-$ for all $t \in M$. Since $[\![\underline{\mu}t]\!] \leqslant \underline{\mu}t < \mu t$, $(t, [\![\underline{\mu}t]\!]) \in A_+$, therefore $\sigma^*(t) = [\![\underline{\mu}t]\!]$ for all $t \in M$. To prove the necessity, it is enough to find a vector $x \in X_{p,q}$, $x \neq 0$, such that all the excesses are strictly negative. In A_-, this is true if

$$\alpha < \max_{(t,\sigma(t))\in A_-} \frac{\lambda\sigma(t) - t}{p\sigma(t)/q - t}.$$

The right side of this inequality is well defined since A_- is a finite set and since $p\sigma(t)/q - t$ is nonzero for $(t, \sigma(t)) \in A_-$. Furthermore this maximum is nonzero since $\lambda\sigma(t) - t \neq 0$ by definition of A_-. In A_+, we show that $p\sigma(t)/q - t \leqslant 0$ for all t. This is true for $t \in M$ since $\sigma^*(t)/t = [\![\underline{\mu}t]\!]/t \leqslant \underline{\mu} = q/p$, and is consequently true for $t \notin M$ since $\sigma^*(t) = [\![\underline{\mu}t]\!]$ in this case by Lemma 4.7. Once we notice that for every coalition $(t, \sigma(t))$ in A_+ such that $p\sigma(t)/q - t = 0$, the excess of this coalition wrt any $x \in X_{p,q}$ is zero, it is clear that the excesses of other coalitions in A_+ will be strictly negative if $x \neq 0$. Finally since $e_x(t, \sigma(t)) = 0$ for all $(t, \sigma(t)) \in A_+$ if $x = ((0)^p, (\lambda)^q)$, this vector cannot be the nucleolus.

(Sufficiency) Let $\lambda \leqslant t^*/[\![\bar{\mu}t^*]\!]$. Then $\sigma^*(t^*) = [\![\bar{\mu}t^*]\!]$. Indeed, $(t^*, [\![\bar{\mu}t^*]\!] + 1) \in A_-$ since $[\![\bar{\mu}t^*]\!] + 1 > \bar{\mu}t^* > \mu t^*$, and $(t^*, [\![\bar{\mu}t^*]\!]) \in A_+$ by hypothesis on λ. Now, $[\![\bar{\mu}t^*]\!]/t^* = ([\![\underline{\mu}t^*]\!] + 1)/t^* > \bar{\underline{\mu}}$, so $p\sigma^*(t^*)/q - t^* > 0$ (the inequality is strict by the definition of the integer part). By Lemma 2.2, we must have $\alpha = 0$. Q.E.D.

The following lemma characterize the coalitions for which the excesses are maximum in A_+ and A_-.

Lemma 4.12: Let $\lambda \in (t^*/[\![\bar{\mu}t^*]\!], p/q]$, then

$$e_{nu}(t^*, [\![\bar{\mu}t^*]\!]) = \max_{(t,\sigma(t))\in A_-} e_{nu}(t, \sigma(t)).$$

Proof: We have shown in the proof of Theorem 4.11 that $p\sigma(t) - t \leqslant 0$ when $(t, \sigma(t)) \in A_+$. Because $\lambda \leqslant p/q$, we deduce that $\{t - \lambda\sigma(t)\}/\{t - p\sigma(t)/q\} \geqslant 1$, and because $1 \geqslant \alpha$, where $x = ((\alpha)^p, (\beta)^q) \in X_{p,q}$, we obtain,

$$\{p\sigma(t)/q - t\}\alpha + t - \lambda\sigma(t) \geqslant 0 \quad \text{if } (t, \sigma(t)) \in A_+. \tag{4.1}$$

Now, let $(t, \sigma(t)) \in A_-$. Then, by Lemma 2.2 and by Theorem 4.11, $e_{nu}(t, \sigma(t)) \leqslant 0$. Consider now two coalitions $(s, \sigma^*(s) + 1)$ and $(t, \sigma^*(t) + 1)$ of A_- and define for

$t > s$ the quantity $\Delta^-(t, s) = e_{nu}(t, \sigma^*(t)+1) - e_{nu}(s, \sigma^*(s)+1)$. It is immediate that $\Delta^-(t, s) = [p\{\sigma^*(t) - \sigma^*(s)\}/q - t + s]\alpha + t - s - \lambda\{\sigma^*(t) - \sigma^*(s)\}$. From (4.1) and the above developments, we can deduce,

$$\Delta^-(t, s) \begin{cases} \geqslant 0 & \text{if } (t-s, \sigma^*(t) - \sigma^*(s)) \in A_+ \\ \leqslant 0 & \text{if } (t-s, \sigma^*(t) - \sigma^*(s)) \in A_-. \end{cases} \tag{4.2}$$

We prove now that for all $s \notin M$, we have $\Delta^-(t_m, s) \geqslant 0$, where $t_m = \min\{t \mid t \in M\}$. If $s < t_m$, then $s \notin M$ and $\sigma^*(t_m - s) = [\![\bar{\mu}(t_m - s)]\!]$. Consequently, $t_m - s > \lambda[\![\bar{\mu}(t_m - s)]\!]$, and $t_m - s > \lambda([\![\bar{\mu}t_m]\!] - [\![\bar{\mu}s]\!] - 1)$ by Lemma 4.10. So, $(t_m - s, \sigma^*(t_m) - \sigma^*(s)) \in A_+$, i.e. $\Delta^-(t_m, s) \geqslant 0$. Consider now $s > t_m$ and $s \notin M$, then, $\sigma^*(s) - \sigma^*(t_m) = [\![\bar{\mu}s]\!] - [\![\bar{\mu}t_m]\!] + 1$. Now, if $s - t_m \in M$, $\sigma^*(s - t_m) = [\![\bar{\mu}(s - t_m)]\!] - 1$ and if $s - t_m \notin M$, $\sigma^*(s - t_m) = [\![\bar{\mu}(s - t_m)]\!]$, so in both cases we have by Lemma 4.10,

$$\begin{aligned} s - t_m &< \lambda([\![\bar{\mu}(s - t_m)]\!] + 1) \\ &\leqslant \lambda([\![\bar{\mu}s]\!] - [\![\bar{\mu}t_m]\!] + 1) \\ &= \lambda(\sigma^*(s) - \sigma^*(t_m)) \end{aligned}$$

and, $\Delta^-(s, t_m) \leqslant 0$ since $(s - t_m, \sigma^*(s) - \sigma^*(t_m)) \in A_-$.

Consequently, if t_0 is the smallest element which satisfies,

$$e_{nu}(t_0, \sigma^*(t_0) + 1) \geqslant e_{nu}(t, \sigma^*(t) + 1) \quad \text{for all } t \neq t_0$$

we must have $t_0 \in M$. Let $t \in M$ and $t > t^*$. By definition of t^*, $t^*/[\![\bar{\mu}t^*]\!] \geqslant t/[\![\bar{\mu}t]\!]$, i.e. $t^*[\![\bar{\mu}t]\!] \geqslant t[\![\bar{\mu}t^*]\!]$. Subtracting from each side $t^*[\![\bar{\mu}t^*]\!]$ and next dividing each side by $[\![\bar{\mu}t^*]\!]([\![\bar{\mu}t]\!] - [\![\bar{\mu}t^*]\!])$ we obtain,

$$t^*/[\![\bar{\mu}t^*]\!] \geqslant (t - t^*)/([\![\bar{\mu}t]\!] - [\![\bar{\mu}t^*]\!])$$

and so, $(t - t^*, \sigma^*(t) - \sigma^*(t^*)) \in A_-$ since $\lambda > t^*/[\![\bar{\mu}t^*]\!]$ and $t, t^* \in M$. Then $\Delta^-(t, t^*) \leqslant 0$ and t_0 must be inferior or equal to t^*. If $t_0 = t^*$, the lemma is proved. Hence, suppose $t_0 \neq t^*$. By definition of t^*, $t^*/[\![\bar{\mu}t^*]\!] > t_0/[\![\bar{\mu}t_0]\!]$ where the inequality is strict when $t_0 < t^*$ (otherwise t^* is not the smallest integer which maximizes $t/[\![\bar{\mu}t]\!]$ for $t \in M$). This inequality is equivalent to the following,

$$(t^* - t_0)/([\![\bar{\mu}t^*]\!] - [\![\bar{\mu}t_0]\!]) > t^*/[\![\bar{\mu}t^*]\!]. \tag{4.3}$$

Because $[\![\bar{\mu}(t^* - t_0)]\!] \leqslant [\![\bar{\mu}t^*]\!] - [\![\bar{\mu}t_0]\!]$ by Lemma 4.10, $t^* - t_0$ is not an element of M. Indeed, if $t^* - t_0 \in M$, then $t^*/[\![\bar{\mu}t^*]\!] > (t^* - t_0)/[\![\bar{\mu}(t^* - t_0)]\!] \geqslant (t^* - t_0)/([\![\bar{\mu}t^*]\!] - [\![\bar{\mu}t_0]\!])$ which contradicts (4.3). Then $t^* - t_0 \notin M$ and we see easily that $[\![\bar{\mu}(t^* - t_0)]\!] = [\![\bar{\mu}t^*]\!] - [\![\bar{\mu}t_0]\!]$. Finally, $(t^* - t_0, [\![\bar{\mu}t^*]\!] - [\![\bar{\mu}t_0]\!]) \in A_+$ which implies by (4.2) that $\Delta^-(t^*, t_0) \geqslant 0$, and so $t^* = t_0$. Q.E.D.

Lemma 4.13: For all t s.t. qt/p is not an integer, we have $\sigma^*(p - t) = q - \sigma^*(t) - 1$.

Proof: We have shown in Lemma 4.7and Theorem 4.11 that $\sigma^*(p-t) = [\![\underline{\mu}(p-t)]\!]$. If $\underline{\mu}t$ is not an integer, we have, $[\![\underline{\mu}(p-t)]\!] = [\![q-\underline{\mu}t]\!] < q - [\![\underline{\mu}t]\!]$. Now, $\underline{\mu}t < \sigma^*(t)+1$ and $q \leqslant \underline{\mu}p$ imply $\underline{\mu}(p-t) > q - \sigma^*(t) - 1$, i.e. $(p-t, q-\sigma^*(t)-1) \in A_+$. It follows that $\sigma^*(p-t) = q - \sigma^*(t) - 1$. We remark that $\underline{\mu}t \notin \mathbb{N}$ for $t \in M$. Q.E.D.

Corollary 4.14:

$$e_{nu}(p-t^*, q-[\![\bar{\mu}t^*]\!]) = \max_{\substack{(t,\sigma(t))\in A_+ \\ qt/p \neq \sigma(t)}} e_{nu}(t, \sigma(t)).$$

Proof: By Lemma 4.13, $\sigma^*(p-t^*) = q - \sigma^*(t^*) - 1 = q - [\![\bar{\mu}t^*]\!]$. By (4.2) and Lemma 4.12, $(t^*-s, \sigma^*(t^*)-\sigma^*(s)) \in A_+$ for all $s < t^*$ and $(s-t^*, \sigma^*(s)-\sigma^*(t^*)) \in A_-$ for all $s > t^*$. If $s < t^*$, $p-s > p-t^*$ and $(p-s)-(p-t^*) = t^*-s$ and $\sigma^*(p-s) - \sigma^*(p-t^*) = \sigma^*(t^*) - \sigma^*(s)$. Let us denote $\Delta^+(t,s) = e_{nu}(t, \sigma^*(t)) - e_{nu}(s, \sigma^*(s))$. Then, $\Delta^+(p-s, p-t^*) = \Delta^+(t^*, s)$. Because $(t^*-s, \sigma^*(t^*)-\sigma^*(s)) \in A_+$ and because the excesses wrt the nucleolus are nonpositive, $\Delta^+(p-s, p-t^*) \leqslant 0$. If $s > t^*$, we have $(s-t^*, \sigma^*(s)-\sigma^*(t^*)) \in A_-$ and so $\Delta^+(p-t^*, p-s) \geqslant 0$ since $p\sigma(t)/q - t > 0$ for $(t, \sigma(t)) \in A_-$. Q.E.D.

Theorem 4.15: If $\lambda \in (t^*/[\![\bar{\mu}t^*]\!], p/q]$,

$$nu = \left(\left(\frac{1}{2}\, \frac{\lambda[\![\bar{\mu}t^*]\!] - t^*}{\frac{p}{q}[\![\bar{\mu}t^*]\!] - t^*} \right)^p, \left(\frac{1}{2}\left(\lambda + \frac{(p-\lambda q)t^*}{p[\![\bar{\mu}t^*]\!] - qt^*} \right) \right)^q \right).$$

Proof: By Lemma 2.4, Lemma 4.12 and Corollary 4.14, the nucleolus is defined by the equality $e_{nu}(t^*, [\![\bar{\mu}t^*]\!]) = e_{nu}(p-t^*, q-[\![\mu t^*]\!])$, i.e. the vector of Theorem 4.15. Q.E.D.

Theorems 4.11 and 4.15 give a complete characterization of the nucleolus when $\lambda \leqslant p/q$. When $\lambda \geqslant p/q$, let $\bar{\mu} = q/p$. Define $\tilde{t} = \min\{t \mid t \in \operatorname{argmin} t/[\![\bar{\mu}t]\!]\}$. Then, by arguments similar to those used in Lemma 4.12, 4.13, it is possible to show that $e_{nu}(\tilde{t}, [\![\bar{\mu}\tilde{t}]\!]) = \max_{(t,\sigma(t))\in A_+} e_{nu}(t, \sigma(t))$ and $e_{nu}(p-\tilde{t}, q-[\![\bar{\mu}\tilde{t}]\!]) = \max_{(t,\sigma(t))\in A_-} e_{nu}(t, \sigma(t))$.

Finally, Lemma 2.4 implies the following.[6]

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[6] Note that argmin $t/[\![\bar{\mu}t]\!]$ is defined over $\{1, \ldots, p-1\}$ here. Note also that when $\lambda \leqslant t/[\![\bar{\mu}t]\!]$, $\sigma^*(t) = [\![\bar{\mu}t]\!]$ for all t. The analysis is consequently simpler here than when $\lambda \leqslant p/q$ since there is no need for defining a set M.

Theorem 4.16: Let $\lambda \geqslant p/q$.

(i) If $\lambda \leqslant \tilde{t}/[\![\bar{\mu}\tilde{t}]\!]$,

$$nu = \left(\left(1 - \frac{q}{2}\,\frac{\tilde{t}-\lambda[\![\bar{\mu}\tilde{t}]\!]}{q\tilde{t}-p[\![\bar{\mu}\tilde{t}]\!]}\right)^p, \left(\frac{p}{2}\,\frac{\tilde{t}-\lambda[\![\bar{\mu}\tilde{t}]\!]}{q\tilde{t}-p[\![\bar{\mu}\tilde{t}]\!]}\right)^q\right).$$

(ii) If $\lambda \geqslant \tilde{t}/[\![\bar{\mu}\tilde{t}]\!], nu = ((1)^p, (0)^q)$.

5 Final Comments

When $q = pk$, $q = pk + 1$ or $q = pk + p - 1$, the nucleolus is given by the formulae of Theorems 3.1 and 4.5. In all other cases, it is possible to use Theorems 4.15 and 4.16 in order to design an algorithm to compute the nucleolus of any game whose characteristic function is given by (1.1) (note that the formulae of Theorem 4.5 are a special case of this algorithm).

For a given λ check first if $\lambda \gtreqless p/q$. Suppose that $\lambda < p/q$ for instance. Next, find the set M of integers verifying $[\![\bar{\mu}t]\!] = [\![\underline{\mu}t]\!] + 1$ where $\bar{\mu}$ and $\underline{\mu}$ are defined by $\bar{\mu} = (q-k)/(p-1)$ and $\underline{\mu} = q/p$. Compute all the ratios $t/[\![\bar{\mu}t]\!]$ for $t \in M$ and select t^*, the smallest integer which maximizes this ratio. The nucleolus is then given by the formulae in Theorem 4.11 or Theorem 4.15 depending upon λ being smaller or greater than $t^*/[\![\bar{\mu}t^*]\!]$. Clearly, even for large values of p and q, the nucleolus can be computed by hand. When $\lambda \geqslant p/q$, it is enough to compute all the ratios $t/[\![\bar{\mu}t]\!]$, where $\bar{\mu} = q/p$ and to determine $\tilde{t}$, the smallest integer for which the ratio $\tilde{t}/[\![\bar{\mu}\tilde{t}]\!]$ is the smallest. The nucleolus is then given by the formula of Theorem 4.16.

While doing the proofs, I have been tempted to conjecture that t^* is in fact the smallest element of M: $t^* = \min\ \{t \mid t \in M\}$. The following example shows that this conjecture is incorrect. Let $p = 19, q = 49$ and $\lambda \in (18/47, 19/49)$. Then, $M = \{5, 10, 12, 15, 17, 18\}$ but the greatest value of $t/[\![\bar{\mu}t]\!]$ is obtained for $t = 12$ and $[\![\bar{\mu} \cdot 12]\!] = 33$.

Finally, if we hold the ratio p/q fixed and if we increase continuously p and q, it is clear that $\bar{\mu} \to \underline{\mu}$, which means that the nucleolus is almost everywhere one of the vectors $((0)^p, (\lambda)^q)$ or $((1)^p, (0)^q)$. In other words, for markets with a large number of traders, all the profit goes to the side with the rare commodity.

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